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TRANSVERSE SHEAR AND NORMAL DEFORMATION HIGHER-ORDER THEORY FOR THE SOLUTION OF DYNAMIC PROBLEMS OF LAMINATED PLATES AND SHELLS

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Abstract—An improved transverse shear and normal deformation higher-order theory is developed for the solution of dynamic problems involving multilayered plates and shells with an arbitrary number and sequence of transversely isotropic layers. The layers may differ significantly in their physical and mechanical properties. The theory developed is based on the kinematic hypotheses which are derived using iterative technique. Dynamic effects, such as forces of inertia, and the direct influence of external loading on the components of stress and strain are included in the initial stage of derivation where kinematic hypotheses are formulated. New variables which have clear physical meanings are introduced. The system of governing differential equations and the complete set of boundary conditions are derived. The closed form solutions are given for problems involving forced and natural vibrations. The numerical results are compared both with three-dimensional solutions, which are available in the literature, and with experimental data. The significant features of the present theory and the implications of the numerical results are discussed.

INTRODUCTION

Recent advances in the technology of composite materials led to the use of composite plates and shells as primary structural components in various engineering applications. These developments necessitate the design of thick laminated composites which are able to sustain higher loading conditions and at the same time combine good weight to strength ratios. Analysis of these structures is complicated by the three-dimensional nature of stress and strain state. It is well known that, due to the anisotropy and heterogeneity of the materials of different layers and the existence of layers which exhibit weak resistance to transverse shear and normal deformations, the classical theory of plates and shells, based on the Kirchhoff–Love hypotheses, leads to substantial errors. The possibility of using a three-dimensional theory is of limited use due to mathematical difficulties and the complexity of laminated systems. As a result, numerous higher-order theories of plates and shells have been formulated in recent years which approximate the three-dimensional solutions with reasonable accuracy. Next, some historical highlights of higher-order theories are discussed.

The investigation of anisotropic shells started in the 1920s and the first recorded paper on this subject was written by Shtayerman (1924). The beginning of the development of refined models for the solution of vibration problems links with the names of Lord Raleigh, who had taken into account the influence of rotary inertia on the normal element of a beam, and Timoshenko (1922), who took into account the influence of transverse shear deformations in the transverse vibration of bars.

Models based on similar assumptions as in Timoshenko (1922) but for the analysis of plates and shells are often called Timoshenko models. A detailed survey of refined theories based on such models was later given by Grygolyuk and Selezov (1972). Important contributions to the development of refined theories were made by Reissner (1945, 1950, 1960). A detailed analysis of Reissner's approach was given by Goldenveizer (1963, 1968) who pointed out the existence of boundary effects due to the influence of transverse shear; this effect is now known as Reissner's boundary effect. Vibration problems for isotropic plates, including the effects of transverse shear deformations and rotary inertia, were solved by Mindlin (1951) who determined the limits of applicability of the Kirchhoff hypotheses. Natural vibrations in sandwich plates and shells, including transverse shear deformations, were investigated by Grygolyuk and Chulkov (1973). The three-dimensional equations of the theory of elasticity for problems involving bending and vibrations of plates and shells were used by Grinchenko (1963), Moskalenko and Novichkov (1968), Noor and Rarig (1974), Grigorenko *et al.* (1974), Noor and Peters (1989), Ren (1989) and Verijenko *et al.* (1993). In recent years, numerous refined approaches for the analysis of composite plates and shells have been formulated. Contributions by Ambartsumyan (1974), Reddy (1979, 1989), Reddy and Phan (1985), Librescu (1987), Vasilyev (1988) and Noor and Peters (1989) should be mentioned. A survey of different theoretical and computational models may be found in reviews by Kant and Junghare (1976), Dudchenko *et al.* (1983), Bert (1984), Noor and Burton (1989, 1990) and Reddy (1990). Several monographs have also been written on the subject (Rikards and Teters, 1974; Librescu, 1975; Bolotin and Novichkov, 1980; Grigorenko and Vasilenko, 1981; Piskunov and Verijenko, 1986; Rasskazov *et al.*, 1986; Bogdanovich, 1987; Piskunov *et al.*, 1987; Whitney, 1987). It is noted that the list of references is not intended to be a comprehensive one and the specific publications were referred to because of their relevance to the present paper.

A study of the literature indicates that, in the case of dynamic analysis of laminated structures in which the layers may have significantly different physical characteristics, it is also necessary to consider the phenomenon of normal deformation. Moreover, most of the known dynamic higher-order theories are based on the hypotheses which are derived from consideration of the quasi-static problem. In this case the kinematic hypotheses do not fully reflect the physical essence of the problem. Therefore, the study of the dynamic behaviour of laminated structures on the basis of improved higher-order theories will fill a gap in the analysis of thick composites under dynamic loads.

The main objective of this paper is to derive a comprehensive higher-order theory of laminated plates and shells which can accurately predict the dynamic behaviour of such structures under various loading conditions, and for a wide range of layer characteristics.

BASIC ASSUMPTIONS AND DERIVATION OF KINEMATIC HYPOTHESES

We consider shells with transversely isotropic layers which are weak in their resistance to transverse shear and normal deformation. No limitations are placed on the thickness, rigidity, density, number and/or sequence of the layers. The physical and mechanical characteristics of the layers may vary through the thickness. The assumption that the layers are perfectly bonded ensures their deformation as a single unit without delamination. Thus, the structure of the shell through the thickness is arbitrarily irregular and heterogeneous. The shell is represented by a curvilinear orthogonal coordinate system x_1Ox_2 which is parallel to the bounding surfaces and surfaces of contact between the layers (Fig. 1). The axes of the curvilinear coordinates $x_i =$ constant ($i = 1, 2$) coincide with the principal lines of curvature and the coordinate $z = x_3$ is defined along the normal to the reference surface x_1Ox_2. It is assumed that the coefficients of the first quadratic form of a surface are close to unity, i.e. $A_1 \approx A_2 \approx 1$, and the main curvatures are constant, i.e. $k_{ij} =$ constant ($i,j = 1,2$). The total thickness of the shell is small in comparison to the radii of the curvatures ($1 + k_{ij} \approx 1$). Dynamic loads are applied on the outer and inner surfaces of the laminate so that

$$p_s^{\pm}(x_i, t) = p^{\pm}, \quad s = 1, 2, 3, \tag{1}$$

where p^+ and p^- are loads applied on the outer and inner surfaces, respectively, and the

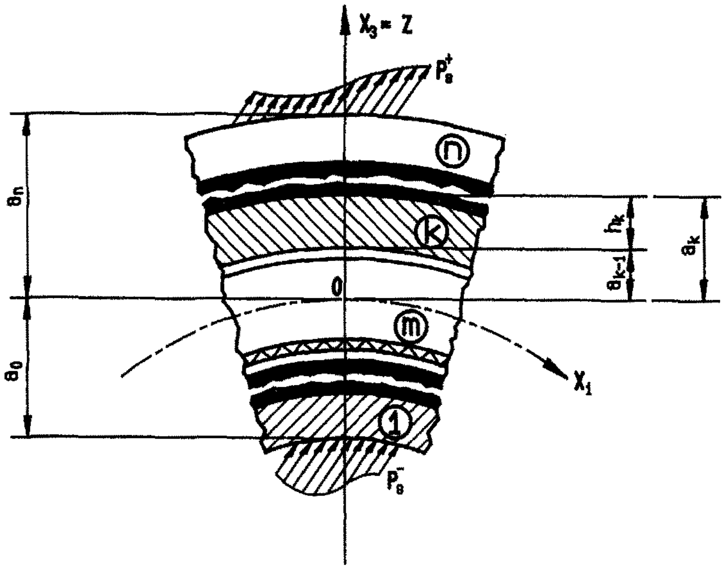


Fig. 1. Geometry of a laminated shell.

subscript s denotes the corresponding coordinate axes. The reference surface x_1Ox_2 may be positioned arbitrarily through the thickness of the shell. It may be chosen, within any layer, to coincide with the interlaminar or external surfaces as dictated by the nature of the problem under consideration. The stress conditions on the external surfaces may be written as

$$\sigma_{s3}^{(1)} = -p_s^- \quad \text{for } z = a_0\,(k=1) \tag{2}$$

$$\sigma_{s3}^{(n)} = +p_s^+ \quad \text{for } z = a_n\,(k=n) \tag{3}$$

$$s = 1,2,3,$$

where k denotes the layer number and n is the total number of layers.

Since the layers are assumed to be perfectly bonded, the continuity conditions for an arbitrary surface $z = a_{k-1}$ are given by

$$\sigma_{s3}^{(k)} = \sigma_{s3}^{(k-1)} \quad \text{(static)} \tag{4}$$

$$u_s^{(k)} = u_s^{(k-1)} \quad \text{(kinematic).} \tag{5}$$

In the following derivations, summation is assumed over subscripts $i,j = 1,2$; $s,r = 1,2,3$, and p,q,f,g. However no summation is implied over $k = 1,2\ldots m\ldots n$. A subscript after a comma denotes differentiation with respect to the variable following the comma and a superscript is expressed in brackets to distinguish it from an exponent.

Considering "small" bending (Novozhilov, 1962), the strain components of the kth layer may be expressed as

$$2e_{ij}^{(k)} = u_{i,j}^{(k)} + u_{j,i}^{(k)} + 2k_{ij}u_3^{(k)}$$

$$2e_{i3}^{(k)} = u_{i,3}^{(k)} + u_{3,i}^{(k)}$$

$$e_{33}^{(k)} = u_{3,3}^{(k)}, \tag{6}$$

where $u_i^{(k)}(x_i, z, t)$ and $u_3^{(k)}(x_i, z, t)$ are displacements of the kth layer in the tangential x_i $(i = 1,2)$ and normal $z = x_3$ directions, respectively, and k_{ij}s are curvatures of the shell. The displacements of the reference surface $(z = 0,\ k = m)$ may be expressed as

$$u_i^{(m)}(x_i, 0, t) = u_i, \quad u_3^{(m)}(x_i, 0, t) = w \tag{7}$$

and the strains and the curvatures due to deformation as

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) + k_{ij} w; \quad \kappa_{ij} = -w_{,ij}, \tag{8}$$

which satisfy the well known relations (Novozhilov, 1962)

$$\begin{aligned} 2\varepsilon_{12,12} - \varepsilon_{11,22} - \varepsilon_{22,11} &= k_{11}\kappa_{22} + k_{22}\kappa_{11} - 2k_{12}\kappa_{12} \\ \kappa_{11,2} - \kappa_{12,1} &= 0, \quad \kappa_{22,1} - \kappa_{12,2} = 0. \end{aligned} \tag{9}$$

The generalized Hooke's law for a transversely isotropic layer k of the shell, where the surface of isotropy at any point (x_i, z) is orthogonal to the normal, may be expressed as (Ambartsumyan, 1974)

$$\begin{aligned} e_{11}^{(k)} &= a_{11}^{(k)}\sigma_{11}^{(k)} + a_{12}^{(k)}\sigma_{22}^{(k)} + a_{13}^{(k)}\sigma_{33}^{(k)} \\ e_{22}^{(k)} &= a_{21}^{(k)}\sigma_{11}^{(k)} + a_{22}^{(k)}\sigma_{22}^{(k)} + a_{23}^{(k)}\sigma_{33}^{(k)} \\ e_{33}^{(k)} &= a_{31}^{(k)}\sigma_{11}^{(k)} + a_{32}^{(k)}\sigma_{22}^{(k)} + a_{33}^{(k)}\sigma_{33}^{(k)} \\ 2e_{23}^{(k)} &= a_{44}^{(k)}\sigma_{23}^{(k)}, \quad 2e_{13}^{(k)} = a_{55}^{(k)}\sigma_{13}^{(k)}, \quad 2e_{12}^{(k)} = a_{66}^{(k)}\sigma_{12}^{(k)}, \end{aligned} \tag{10}$$

where $a_{ij}^{(k)}$ denotes the elastic compliance coefficients of the kth layer. In eqn (10) the compliance characteristics are given by

$$a_{11}^{(k)} = a_{22}^{(k)} = \frac{1}{E_k}, \quad a_{33}^{(k)} = \frac{1}{E'_k}, \quad a_{12}^{(k)} = a_{21}^{(k)} = -\frac{\nu_k}{E_k}$$

$$a_{13}^{(k)} = a_{31}^{(k)} = a_{23}^{(k)} = a_{32}^{(k)} = -\frac{\nu'_k}{E'_k}, \quad a_{44}^{(k)} = a_{55}^{(k)} = \frac{1}{G'_k}, \quad a_{66}^{(k)} = \frac{1}{G_k},$$

where $E_k = E_k(z)$, $\nu_k = \nu_k(z)$, $G_k = G_k(z)$ are the modulus of elasticity, Poisson's ratio and shear modulus in the plane of isotropy, respectively; $E'_k = E'_{k(}(z)$ and $G'_k = G'_k(z)$ are moduli of elasticity and shear in the transversal direction; $\nu'_k = \nu'_k(z)$ is Poisson's ratio, which characterizes reduction in the plane of isotropy when tension is applied in the transversal direction. All elastic properties of the kth layer are assumed to be functions of the coordinate z $(a_{k-1} \leqslant z \leqslant a_k)$. The Hooke's law for specific cases of material can be obtained by specifying the material properties. For example, if it is assumed that $\nu'_k = 0$, then $a_{13}^{(k)} = a_{31}^{(k)} = a_{23}^{(k)} = a_{32}^{(k)} = 0$ and we have a layer for which the influence of the normal stresses $\sigma_{33}^{(k)}$ on the tangential components $e_{11}^{(k)}, e_{22}^{(k)}$ of the strain tensor is excluded and, similarly, the influence of the tangential components $\sigma_{11}^{(k)}, \sigma_{22}^{(k)}$ of the stress tensor on the strain $e_{33}^{(k)}$ in the transversal direction is also excluded. Thus the normal deformation due to the Poisson's effect is excluded. If $E'_k = \infty$, then $a_{13}^{(k)} = a_{31}^{(k)} = a_{23}^{(k)} = a_{32}^{(k)} = a_{33}^{(k)} = 0$ and we have an incompressible layer $(e_{33}^{(k)} = 0)$ in which the strains $e_{11}^{(k)}, e_{22}^{(k)}$ are independent of the normal stress $\sigma_{33}^{(k)}$. This effect may be obtained by assuming $\sigma_{33}^{(k)} = 0$. Furthermore, if we assume $G'_k = \infty$, then the layer also becomes perfectly rigid under the transversal shear, i.e. $e_{13}^{(k)} = e_{23}^{(k)} = 0$. The above set of assumptions is equivalent to that used in the derivation of the classical theory which is a special case of the theory presented in this study.

Classical model

We first obtain the expressions of the classical model based on the Kirchhoff–Love hypotheses. We will subsequently make use of them for the derivation of kinematic hypotheses of the higher-order theory. In the classical model the following relations can be written for the kth layer:

$$e_{i3}^{(k)} = 0, \quad e_{33}^{(k)} = 0, \quad \sigma_{33}^{(k)} = 0, \quad i = 1, 2. \tag{11}$$

Substituting eqns (6) into the first two hypotheses (11), and integrating the resulting expressions, the kinematic model of the shell may be obtained as

$$u_i^{(k)} = u_i - w_{,i}z, \quad u_3^{(k)} = w. \tag{12}$$

In these calculations, the continuity conditions (5) and relations (7) have been taken into account. The strains of the kth layer in tangential directions can be obtained from eqns (6) and (12) as

$$e_{ij}^{(k)} = \varepsilon_{ij} + \kappa_{ij}z, \quad i,j = 1,2, \tag{13}$$

where $\varepsilon_{12} = \varepsilon_{21}, \kappa_{12} = \kappa_{21}, e_{12}^{(k)} = e_{21}^{(k)}$.

The normal stresses in the kth layer may be determined from eqns (10) in conjunction with the static hypothesis $\sigma_{33}^{(k)} = 0$, or from assumption $E_k' = \infty$ together with eqn (13). These calculations give

$$\sigma_{11}^{(k)} = E_{0k}[(\varepsilon_{11} + \nu_k\varepsilon_{22}) + (\kappa_{11} + \nu_k\kappa_{22})z], \quad \sigma_{22}^{(k)} \rightleftharpoons \sigma_{11}^{(k)},$$
$$\sigma_{12}^{(k)} = E_{0k}(1-\nu_k)(\varepsilon_{12} + \kappa_{12}z) \tag{14}$$

where

$$E_{0k} = \frac{E_k}{1-\nu_k^2}, \quad 2G_k = \frac{E_k}{1+\nu_k} = E_{0k}(1-\nu_k).$$

In eqn (14), the symbol $\rightleftharpoons$ indicates that the expression for $\sigma_{22}^{(k)}$ is of the same form as that for $\sigma_{11}^{(k)}$ with the provision that the subscript 11 is replaced with 22 and vice versa. The transverse shear and normal stresses cannot be found from the Hooke's law because of the hypotheses (11). In order to determine these stresses, we use the equations of motion of the shell, and for the kth layer they may be written as (Novozhilov, 1962)

$$\sigma_{ij,j}^{(k)} + \sigma_{i3,3}^{(k)} = \rho_k\ddot{u}_i^{(k)}, \quad \sigma_{33,3}^{(k)} + \sigma_{i3,i}^{(k)} - k_{ij}\sigma_{ij}^{(k)} = \rho_k\ddot{u}_3^{(k)}, \quad i,j = 1, 2, \tag{15}$$

where $\rho_k\ddot{u}_i^{(k)} = \rho_k u_{i,tt}^{(k)}, \rho_k\ddot{u}_3^{(k)} = \rho_k u_{3,tt}^{(k)}$ are the forces of inertia in the tangential and normal directions and $\rho_k = \rho_k(z)$ is the material density of the kth layer. From the first expression in eqn (15), we obtain the transverse shear stresses as

$$\sigma_{i3}^{(k)} = -\int_{a_{k-1}}^{z} (\sigma_{ij,j}^{(k)} - \rho_k\ddot{u}_i^{(k)})\,dz + \Phi_{ik} \tag{16}$$

and, using the second expression in (15), we derive the transverse normal stress as

$$\sigma_{33}^{(k)} = -\int_{a_{k-1}}^{z} (\sigma_{i3,i}^{(k)} - k_{ij}\sigma_{ij}^{(k)} - \rho_k\ddot{u}_3^{(k)})\,dz + \Phi_{3k}, \tag{17}$$

where Φ_{ik} are the functions of integration for the kth layer. The functions of integrations Φ_{ik} are determined using condition (2) by setting $s = i$ so that

$$\Phi_{ik} = -p_i^- - \int_{a_0}^{a_{k-1}} (\sigma_{ij,j}^{(k)} - \rho_k \ddot{u}_i^{(k)})\,dz. \tag{18}$$

Here the following rules of integration for piecewise functions and for integration with variable upper limits have been used:

$$\int_{a_0}^{a_{k-1}} (\ldots)^{(k)}\,dz = \sum_{r=1}^{k-1} \int_{a_{r-1}}^{a_r} (\ldots)^{(r)}\,dz$$

$$\int_{a_0}^{z} (\ldots)^k\,dz = \int_{a_{k-1}}^{z} (\ldots)^k\,dz + \sum_{r=1}^{k-1} \int_{a_{r-1}}^{a_r} (\ldots)^r\,dz.$$

Substituting eqn (18) into eqn (16), we obtain the transverse shear stresses

$$\sigma_{i3}^{(k)} = -p_i^- - \int_{a_0}^{z} (\sigma_{ij,j}^{(k)} - \rho_k \ddot{u}_i^{(k)})\,dz. \tag{19}$$

The expression for the external loading may be found using condition (3) in eqn (19) with $s = i$:

$$p_i^- + p_i^+ = -\int_{a_0}^{a_n} (\sigma_{ij,j}^{(k)} - \rho_k \ddot{u}_i^{(k)})\,dz. \tag{20}$$

The tangential forces and forces of inertia may now be determined as

$$N_{ij} = \int_{a_0}^{a_n} \sigma_{ij}^{(k)}\,dz, \quad i,j = 1,2 \tag{21}$$

$$T_i = \int_{a_0}^{a_n} \rho_k \ddot{u}_i^{(k)}\,dz, \quad i = 1,2. \tag{22}$$

From eqn (20) we may obtain the well known equations of motion for the classical theory of shells as

$$N_{ij,j} - T_i + p_i^+ + p_i^- = 0, \quad i,j = 1,2. \tag{23}$$

The transverse normal stresses may be found from eqn (17) in conjunction with condition (2) and are given by

$$\sigma_{33}^{(k)} = -p_3^- - \int_{a_0}^{z} [\sigma_{i3,i}^{(k)} - k_{ij}\sigma_{ij}^{(k)} - \rho_k \ddot{u}_3^{(k)}]\,dz. \tag{24}$$

Moreover,

$$p_3^- + p_3^+ = -\int_{a_0}^{a_n} [\sigma_{i3,i}^{(k)} - k_{ij}\sigma_{ij}^{(k)} - \rho_k \ddot{u}_3^{(k)}]\,dz. \tag{25}$$

The third equation of motion for the classical theory of shells may now be written as

$$M_{ij,ij}-k_{ij}N_{ij}-(T_{ii}+T_3)+(p_3^++p_3^-)+(p_{i,i}^+a_n+p_{i,i}^-a_0)=0,\quad i,j=1,2, \tag{26}$$

where

$$M_{ij}=\int_{a_0}^{a_n}\sigma_{ij}^{(k)}z\,\mathrm{d}z,\quad i,j=1,2 \tag{27}$$

are the moments of the internal forces. The moments T_{ij}, due to the forces of inertia which are acting in the tangential directions, are given by

$$T_{ij}=\int_{a_0}^{a_n}\rho_k\ddot{u}_{i,j}^{(k)}z\,\mathrm{d}z,\quad i=1,2,\quad j=i \tag{28}$$

and the force of inertia in the normal direction is given as

$$T_3=\int_{a_0}^{a_n}\rho_k\ddot{u}_3^{(k)}\,\mathrm{d}z. \tag{29}$$

The eqns (23) and (26) represent the system of equations of motion for the classical theory of laminated shells. Using eqns (21), (27) and expressions (12)–(14) we can rewrite this system in terms of displacements u_i and w. Next we use the above expressions in the derivation of the kinematic hypotheses of the higher-order theory.

Transverse shear stresses

Using eqns (13) and (14) in conjunction with eqns (19) and (20), we can obtain the following relations for the case $i=1$:

$$\begin{aligned}\sigma_{13}^{(k)}&=-p_1^--\int_{a_0}^{z}(\sigma_{11,1}^{(k)}+\sigma_{12,2}^{(k)}-\rho_k\ddot{u}_1^{(k)})\,\mathrm{d}z\\&=-p_1^--\left[(\varepsilon_{11,1}+\varepsilon_{12,2})\int_{a_0}^{z}E_{0k}\,\mathrm{d}z+(\varepsilon_{22,1}-\varepsilon_{12,2})\int_{a_0}^{z}E_{0k}\nu_k\,\mathrm{d}z\right]\\&\quad-\left[(\kappa_{11,1}+\kappa_{12,2})\int_{a_0}^{z}E_{0k}z\,\mathrm{d}z+(\kappa_{22,1}-\kappa_{12,2})\int_{a_0}^{z}E_{0k}\nu_kz\,\mathrm{d}z\right]+\left(\ddot{u}_1\int_{a_0}^{z}\rho_k\,\mathrm{d}z-\ddot{w}_{,1}\int_{a_0}^{z}\rho_kz\,\mathrm{d}z\right)\end{aligned} \tag{30}$$

$$\begin{aligned}p_1^++p_1^-&=-\left[(\varepsilon_{11,1}+\varepsilon_{12,2})\int_{a_0}^{a_n}E_{0k}\,\mathrm{d}z+(\varepsilon_{22,1}-\varepsilon_{12,2})\int_{a_0}^{a_n}E_{0k}\nu_k\,\mathrm{d}z\right]\\&\quad-\left[(\kappa_{11,1}+\kappa_{12,2})\int_{a_0}^{a_n}E_{0k}z\,\mathrm{d}z+(\kappa_{22,1}-\kappa_{12,2})\int_{a_0}^{a_n}E_{0k}\nu_kz\,\mathrm{d}z\right]+\left(\ddot{u}_1\int_{a_0}^{a_n}\rho_k\,\mathrm{d}z-\ddot{w}_{,1}\int_{a_0}^{a_n}\rho_kz\,\mathrm{d}z\right).\end{aligned} \tag{31}$$

We introduce through the thickness distribution functions given by

$$f_k(z)=\int_{a_0}^{z}E_{0k}\,\mathrm{d}z,\quad f_{\nu k}=\int_{a_0}^{z}E_{0k}\nu_k\,\mathrm{d}z$$

$$\bar{f}_k(z)=\int_{a_0}^{z}E_{0k}z\,\mathrm{d}z,\quad \bar{f}_{\nu k}(z)=\int_{a_0}^{z}E_{0k}\nu_kz\,\mathrm{d}z$$

$$f_{\rho k}(z) = \int_{a_0}^{z} \rho_k \, dz, \quad \bar{f}_{\rho k}(z) = \int_{a_0}^{z} \rho_k z \, dz. \tag{32}$$

Similarly we introduce the constants

$$B = \int_{a_0}^{a_n} E_{0k} \, dz = \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} E_{0k} \, dz, \quad B_v = \int_{a_0}^{a_n} E_{0k} v_k \, dz$$

$$B_1 = \int_{a_0}^{a_n} E_{0k} z \, dz, \quad B_{1v} = \int_{a_0}^{a_n} E_{0k} v_k z \, dz$$

$$B_\rho = \int_{a_0}^{a_n} \rho_k \, dz, \quad B_{1\rho} = \int_{a_0}^{a_n} \rho_k z \, dz. \tag{33}$$

We note the following relations in eqns (30) and (31):

$$\kappa_{11,1} + \kappa_{12,2} = -(w_{,11} + w_{,22})_{,1} = -\Delta w_{,1} \tag{34}$$

$$\kappa_{22,1} - \kappa_{12,2} = (w_{,22} - w_{,22})_{,1} = 0.$$

Then, substituting eqns (32)–(34) into eqns (30) and (31) gives

$$\sigma_{13}^{(k)} = -p_1 - [(\varepsilon_{11,1} + \varepsilon_{12,2}) f_k + (\varepsilon_{22,1} - \varepsilon_{12,2}) f_{vk}] + \Delta w_{,1} \bar{f}_k + \ddot{u}_1 f_{\rho k} - \ddot{w}_{,1} \bar{f}_{\rho k} \tag{35}$$

$$p_1^+ + p_1^- = -[(\varepsilon_{11,1} + \varepsilon_{12,2}) B + (\varepsilon_{22,1} - \varepsilon_{12,2}) B_v] + \Delta w_{,1} B_1 + \ddot{u}_1 B_\rho - \ddot{w}_{,1} B_{1\rho}. \tag{36}$$

Now we eliminate from eqn (35) those terms which contain the tangential strains ε_{11}, ε_{22} and ε_{12}. In order to do this we rewrite eqn (36) as

$$[(\varepsilon_{11,1} + \varepsilon_{12,2}) + (\varepsilon_{22,1} - \varepsilon_{12,2}) v] = -\frac{p_1^- + p_1^+}{B} + \Delta w_{,1} \frac{B_1}{B} + \ddot{u}_1 \frac{B_\rho}{B} - \ddot{w}_{,1} \frac{B_{1\rho}}{B}, \tag{37}$$

where v is a generalized Poisson's ratio for the entire shell thickness which is taken to be equal for each layer, that is,

$$v = v_k = \frac{B_v}{B}. \tag{38}$$

Here v represents an average Poisson's ratio for the reference surface. For individual layers the exact values of the Poisson's ratios are used, as shown in eqn (33). By doing this the error introduced by taking an average v for the reference surface has a minimal effect on the overall results. Then, taking into account that $f_{vk} = v f_k$, we can write

$$(\varepsilon_{11,1} + \varepsilon_{12,2}) f_k + (\varepsilon_{22,1} - \varepsilon_{12,2}) f_{vk} = [(\varepsilon_{11,1} + \varepsilon_{12,2}) + (\varepsilon_{22,1} - \varepsilon_{12,2}) v] f_k. \tag{39}$$

Substituting eqn (39) into eqn (35) and taking into account eqn (37), we obtain the transverse shear stresses as

$$\sigma_{13}^{(k)} = \Delta w_{,1} \left(\bar{f}_k - \frac{B_1}{B} f_k \right) + p_1^- \left(\frac{f_k}{B} - 1 \right) + p_1^+ \frac{f_k}{B} + \ddot{u}_1 \left(f_{\rho k} - \frac{B_\rho}{B} f_k \right) - \ddot{w}_{,1} \left(\bar{f}_{\rho k} - \frac{B_{1\rho}}{B} f_k \right). \tag{40}$$

The expression for $\sigma_{23}^{(k)}$ can be obtained in a similar manner. The general expression for the transverse shear stresses ($i = 1, 2$) may be written as

$$\sigma_{i3}^{(k)} = \Delta w_{,i} f_{1k} + p_i^- f_{2k} + p_i^+ f_{3k} + \ddot{u}_i f_{4k} - \ddot{w}_{,i} f_{5k}, \tag{41}$$

where the distribution functions are given by

$$f_{1k}(z) = \bar{f}_k - \frac{B_1}{B} f_k, \quad f_{2k}(z) = \frac{f_k}{B} - 1, \quad f_{3k}(z) = \frac{f_k}{B}$$

$$f_{4k}(z) = f_{\rho k} - \frac{B_\rho}{B} f_k, \quad f_{5k}(z) = \bar{f}_{\rho k} - \frac{B_{1\rho}}{B} f_k. \tag{42}$$

Expression (41) differs from those given in Piskunov *et al.* (1987, 1993) as it contains terms which take into account the influence of the tangential forces of inertia and rotary inertia.

Transverse normal stresses

Substituting eqn (12) in eqn (24) and using expression (41) we obtain the transverse normal stresses in the following form:

$$\sigma_{33}^{(k)} = -p_3^- - \Delta w_{,ii} \int_{a_0}^{z} f_{1k}\,dz - p_{i,i}^- \int_{a_0}^{z} f_{2k}\,dz - p_{i,i}^+ \int_{a_0}^{z} f_{3k}\,dz$$
$$- \ddot{u}_{i,i} \int_{a_0}^{z} f_{4k}\,dz + \ddot{w}_{,ii} \int_{a_0}^{z} f_{5k}\,dz + \ddot{w} f_{\rho k} + k_{ij} \int_{a_0}^{z} \sigma_{ij}^{(k)}\,dz, \quad i,j = 1,2. \tag{43}$$

Using eqn (14) we may rewrite the last term in eqn (43) as

$$k_{ij} \int_{a_0}^{z} \sigma_{ij}^{(k)}\,dz = k_\varepsilon f_k + k_\kappa \bar{f}_k \tag{44}$$

where

$$k_\varepsilon = k_{ij}\varepsilon_{ij} + \nu(k_{11}\varepsilon_{22} - 2k_{12}\varepsilon_{12} + k_{22}\varepsilon_{11})$$
$$k_\kappa = k_{ij}\kappa_{ij} + \nu(k_{11}\kappa_{22} - 2k_{12}\kappa_{12} + k_{22}\kappa_{11}). \tag{45}$$

Let us also rewrite eqn (25) as

$$p_3^- + p_3^+ = -\Delta w_{,ii} D_1 - p_{i,i}^- D_2 - p_{i,i}^+ D_3 - \ddot{u}_{i,i} D_4 - \ddot{w}_{,ii} D_5 + \ddot{w} D_6 + k_\varepsilon D_7 + k_\kappa D_8, \tag{46}$$

where the following integral stiffnesses were introduced:

$$D_r = \int_{a_0}^{a_n} f_{rk}\,dz, \quad r = 1\ldots5, \quad D_6 = B_\rho, \quad D_7 = B, \quad D_8 = B_1. \tag{47}$$

Equation (46) may be solved for the operator $\Delta w_{,ii} = \Delta^2 w$, and using the result thus obtained we can eliminate this operator from eqn (43). Then the transverse normal stress may be written in the final form as

$$\sigma_{33}^{(k)} = p_3^- F_{1k} + p_3^+ F_{2k} + p_{i,i}^- F_{3k} + p_{i,i}^+ F_{4k} + \ddot{u}_{i,i} F_{5k} - \Delta \ddot{w} F_{6k} - \ddot{w} F_{7k} - k_\varepsilon F_{8k} - k_\kappa F_{9k} \tag{48}$$

where

$$F_k = \int_{a_0}^{z} f_{1k}\,dz, \quad F_{1k} = \frac{F_k}{D_1} - 1, \quad F_{2k} = \frac{F_k}{D_1}$$

$$F_{(r+1)k} = \frac{D_r}{D_1} F_k - \int_{a_0}^{z} f_{rk}\,dz, \quad r = 2\ldots 5 \tag{49}$$

$$F_{7k} = \frac{D_6}{D_1} F_k - f_{\rho k}, \quad F_{8k} = \frac{D_7}{D_1} F_k - \int_{a_0}^{z} f_k\,dz, \quad F_{9k} = \frac{D_8}{D_1} F_k - \int_{a_0}^{z} \bar{f}_k\,dz. \tag{50}$$

The expression for the transverse normal stresses includes terms which take into account the influence of the inertia forces.

Transverse deformations

Using eqn (41) the transverse shear strains can be obtained from the Hooke's law as

$$2e_{i3}^{(k)} = \frac{\sigma_{i3}^{(k)}}{G_k'} = \Delta w_{,i} \bar{f}_{1k} + p_i^- \bar{f}_{2k} + p_i^+ \bar{f}_{3k} + \ddot{u}_i \bar{f}_{4k} - \ddot{w}_{,i} \bar{f}_{5k}, \tag{51}$$

where the through the thickness distribution functions of the transverse shear are defined as

$$\bar{f}_{rk}(z) = \frac{f_{rk}(z)}{G_k'}, \quad r = 1\ldots 5. \tag{52}$$

Using eqns (10) and (48) we can find the normal strains as

$$\begin{aligned} e_{33}^{(k)} &= -\frac{\nu_k'}{E_k'}(\sigma_{11}^{(k)} + \sigma_{22}^{(k)}) + \frac{\sigma_{33}^{(k)}}{E_k'} \\ &= \nu_{0k}' z \Delta w - \nu_{0k}' u_{i,i} + p_3^- \bar{F}_{1k} + p_3^+ \bar{F}_{2k} + p_{i,i}^- \bar{F}_{3k} + p_{i,i}^+ \bar{F}_{4k} \\ &\quad + \ddot{u}_{i,i} \bar{F}_{5k} - \Delta \ddot{w}\, \bar{F}_{6k} - \ddot{w}\, \bar{F}_{7k} - k_\varepsilon \bar{F}_{8k} - k_\kappa \bar{F}_{9k}, \end{aligned} \tag{53}$$

where the generalized Poisson's ratio of the material of the kth layer is defined as

$$\nu_{0k}' = \frac{E_k \nu_k'}{E_k'(1 - \nu_k)} \tag{54}$$

and we also define

$$\bar{F}_{rk}(z) = \frac{F_{rk}}{E_k'}, \quad r = 1\ldots 9. \tag{55}$$

The above expressions for transverse shear and normal strains as well as expressions for corresponding stresses are not relevant in the classical theory since they only demonstrate the contradictions in this theory. However, they are important for the derivation of the higher-order theory which follows.

DERIVATION OF THE HIGHER-ORDER THEORY

Hypotheses

In deriving a higher-order theory we assume that the transverse shear and normal strains as well as the transverse normal stresses are not equal to zero, that is

$$e_{i3}^{(k)} \neq 0, \quad e_{33}^{(k)} \neq 0, \quad \sigma_{33}^{(k)} \neq 0. \tag{56}$$

These quantities can be expressed using eqns (48), (51) and (53). Using the expressions for the strains and the strain–displacement relations (6), we can find the more accurate components of the displacement vector which constitutes the next stage of the derivation.

Normal displacements

Integrating the third equation in (6) we obtain

$$u_3^{(k)} = w + \int_0^z e_{33}^{(k)}\,dz, \tag{57}$$

where $w = u_3^{(m)}(x_i, 0, t)$ is the normal displacement of the reference surface positioned arbitrarily through the thickness of layers ($k = m$). Substituting expression (54) for $e_{33}^{(k)}$ into (57) and satisfying conditions (5) with $s = 3$, we introduce the following distribution functions of the normal component of the displacement vector:

$$\begin{aligned}
&\varphi_{11}^{(k)} = 1, && \varphi_{21}^{(k)} = \int_0^z v'_{0k} z\,dz, && \varphi_{31}^{(k)} = -\int_0^z v'_{0k}\,dz, \\
&\varphi_{12}^{(k)} = -\int_0^z \bar{F}_{7k}\,dz, && \varphi_{22}^{(k)} = -\int_0^z \bar{F}_{6k}\,dz, && \varphi_{32}^{(k)} = \int_0^z \bar{F}_{5k}\,dz \\
&\varphi_{13}^{(k)} = \int_0^z \bar{F}_{1k}\,dz, && \varphi_{23}^{(k)} = \int_0^z \bar{F}_{3k}\,dz, && \varphi_{33}^{(k)} = -\int_0^z \bar{F}_{8k}\,dz \\
&\varphi_{14}^{(k)} = \int_0^z \bar{F}_{2k}\,dz, && \varphi_{24}^{(k)} = \int_0^z \bar{F}_{4k}\,dz, && \varphi_{34}^{(k)} = -\int_0^z \bar{F}_{9k}\,dz.
\end{aligned} \tag{58}$$

The equation for the normal displacements may now be written as

$$\begin{aligned}
u_3^{(k)} = {} & w\varphi_{11}^{(k)} + \Delta w\varphi_{21}^{(k)} + u_{i,i}\varphi_{31}^{(k)} + \ddot{w}\,\varphi_{12}^{(k)} + \Delta\ddot{w}\,\varphi_{22}^{(k)} + \ddot{u}_{i,i}\varphi_{32}^{(k)} \\
& + p_3^-\varphi_{13}^{(k)} + p_{i,i}^-\varphi_{23}^{(k)} + k_\varepsilon\varphi_{33}^{(k)} + p_3^+\varphi_{14}^{(k)} + p_{i,i}^+\varphi_{24}^{(k)} + k_\kappa\varphi_{34}^{(k)}, \quad i = 1, 2.
\end{aligned} \tag{59}$$

The distribution functions of the normal displacement in eqn (59) allow us to satisfy the continuity conditions on the layer interfaces for the normal displacement when the reference surface is positioned arbitrarily through the thickness of layers.

Tangential displacements

From the second expression in eqn (6) we obtain

$$u_{i,3}^{(k)} = 2e_{i3}^{(k)} - u_{3,i}^{(k)} \tag{60}$$

and integrating this relation we have

$$u_i^{(k)} = u_i + \int_0^z (2e_{i3}^{(k)} - u_{3,i}^{(k)})\,dz, \tag{61}$$

where $u_i = u_i^{(m)}(x_i, 0, t)$ are the tangential displacements of the reference surface. We introduce the distribution functions given by

$$\psi_1^{(k)} = 1, \qquad \psi_{11}^{(k)} = \int_0^z \varphi_{11}^{(k)}\,dz$$

$$\psi_{21}^{(k)} = \int_0^z (\varphi_{21}^{(k)} - \bar{f}_{1k}\,dz, \quad \psi_{31}^{(k)} = \int_0^z \varphi_{31}^{(k)}\,dz$$

$$\psi_2^{(k)} = \int_0^z \bar{f}_{4k}\,dz, \qquad \psi_{12}^{(k)} = \int_0^z (\varphi_{12}^{(k)} + \bar{f}_{5k})\,dz$$

$$\psi_{22}^{(k)} = \int_0^z \varphi_{22}^{(k)}\,dz, \qquad \psi_{32}^{(k)} = \int_0^z \varphi_{32}^{(k)}\,dz$$

$$\psi_3^{(k)} = -\int_0^z \bar{f}_{2k}\,dz, \qquad \psi_4^{(k)} = -\int_0^z \bar{f}_{3k}\,dz, \quad \psi_{pg}^{(k)} = \int_0^z \varphi_{pg}^{(k)}\,dz$$

$$p = 1,2,3, \qquad g = 3,4. \tag{62}$$

Substituting eqn (51) in eqn (61), satisfying conditions (5) ($s = i = 1,2$) and using the functions defined in eqn (62), the expression for tangential displacements may be written as

$$u_i^{(k)} = u_i\psi_1^{(k)} - w_{,i}\psi_{11}^{(k)} - \Delta w_{,i}\psi_{21}^{(k)} - u_{i,ij}\psi_{31}^{(k)} + \ddot{u}_i\psi_2^{(k)} - \ddot{w}_{,i}\psi_{12}^{(k)} - \Delta\ddot{w}_{,i}\psi_{22}^{(k)} - \ddot{u}_{i,ij}\psi_{32}^{(k)}$$
$$+ p_i^-\psi_3^{(k)} - p_{3,i}^-\psi_{13}^{(k)} - p_{i,ij}^-\psi_{23}^{(k)} - k_\varepsilon\psi_{24}^{(k)} + p_i^+\psi_4^{(k)} - p_{3,i}^+\psi_{14}^{(k)} - p_{i,ij}^+\psi_{24}^{(k)} - k_\kappa\psi_{34}^{(k)}, \quad i,j = 1,2. \tag{63}$$

As was the case for normal displacements, the distribution functions defined in eqn (62) allow us to satisfy the continuity conditions in between the layers for the tangential displacements when the reference surface is positioned arbitrarily through the thickness of the shell.

Relations for the higher-order theory

Expressions (59) and (63) for the displacements are written in terms of the unknown functions u_i and w of the classical theory. In order to derive a nonclassical higher-order theory, we introduce new unknown functions of the reference surface using the following irreversible relations:

$$[u_i; w; \Delta w; u_{i,j}] \to [v_{i1}; \chi_{11}; \chi_{21}; \chi_{31}] \tag{64a}$$

$$[\ddot{u}_i; \ddot{w}; \Delta\ddot{w}; \ddot{u}_{i,j}] \to [v_{i2}; \chi_{12}; \chi_{22}; \chi_{32}], \quad \text{i, j} = 1,2. \tag{64b}$$

The physical meanings of the new unknown functions defined in eqn (64) may be deduced from eqns (59) and (63) and were explained in detail in Piskunov *et al.*, (1993). Briefly, these functions describe behavior of the normal which is distorted due to the influence of the transverse shear and the normal deformation. We call them shear and compression functions. Complementary to the functions in eqn (64a), which were introduced in Piskunov *et al.* (1993), in the case of a dynamic problem we also have dynamic shear and dynamic compression functions which are given by expression (64b).

Let us introduce the following relations for the functions of the given external loading conditions, their derivatives and also components which depend on the curvature of the shell. These relations are defined as

$$[p_i^-; p_3^-; p_{i,j}^-; k_\varepsilon] = [v_{i3}; \chi_{13}; \chi_{23}; \chi_{33}]$$

$$[p_i^+; p_3^+; p_{i,j}^+; k_\kappa] = [v_{i4}; \chi_{14}; \chi_{24}; \chi_{34}], \quad i,j = 1,2. \tag{65}$$

These functions are known and they are determined from the solution obtained using the classical theory and from the given loadings p^- and p^+ on the external surfaces and, therefore, take into account directly the effects of the transverse shear and normal deformation due to the loading conditions.

Replacing the functions in eqns (59) and (63) in accordance with the relationships defined by eqns (64) and (65), the expressions for the components of the displacement vector may be written as

$$u_i^{(k)} = v_{ig}\psi_{0g}^{(k)} - \chi_{pg,i}\psi_{pg}^{(k)} \tag{66}$$

$$u_3^{(k)} = \chi_{pg}\varphi_{pg}^{(k)}, \quad i = 1,2; \quad p = 1,2,3; \quad g = 1,2,3,4. \tag{67}$$

The expressions (66) and (67) will be used as kinematic hypotheses for the derivation of the higher-order theory.

Let us now obtain the components of the strain tensor for the kth layer. Taking into account the kinematic hypotheses (66) and (67), the tangential components may be written as

$$e_{ij}^{(k)} = \tfrac{1}{2}(u_{i,j}^{(k)} + u_{j,i}^{(k)}) + k_{ij}u_3^{(k)} = \tfrac{1}{2}[(v_{ig,j} + v_{jg,i})\psi_g^{(k)} - (\chi_{pg,ij} + \chi_{pg,ji})\psi_{pg}^{(k)}] + k_{ij}\chi_{pg}\varphi_{pg}^{(k)}$$
$$i,j = 1,2; \quad p = 1,2,3; \quad g = 1,2,3,4. \tag{68}$$

and the transverse shear strains as

$$2e_{i3}^{(k)} = u_{i,3}^{(k)} + u_{3,i}^{(k)} = v_{ig}\alpha_g^{(k)} + \chi_{pg,i}\alpha_{pg}^{(k)}, \quad i = 1,2; \quad p = 1,2,3; \quad g = 1,2,3,4 \tag{69}$$

where the following notations are employed:

$$\alpha_g^{(k)} = \psi_{g,3}^{(k)}, \quad \alpha_{pg}^{(k)} = \varphi_{pg}^{(k)} - \psi_{pg,3}^{(k)}. \tag{70}$$

The strains due to normal compression are given by

$$e_{33}^{(k)} = u_{3,3}^{(k)} = \chi_{pg}^{(k)}\beta_{pg}^{(k)}, \quad p = 1,2,3; \quad g = 1,2,3,4 \tag{71}$$

where

$$\beta_{pg}^{(k)} = \varphi_{pg,3}^{(k)}. \tag{72}$$

The components of the stress tensor can be determined using the Hooke's law for a transversely isotropic material which are

$$\sigma_{11}^{(k)} = A_{11}^{(k)}e_{11}^{(k)} + A_{12}^{(k)}e_{22}^{(k)} + A_{13}^{(k)}e_{33}^{(k)}$$
$$\sigma_{22}^{(k)} = A_{21}^{(k)}e_{11}^{(k)} + A_{22}^{(k)}e_{22}^{(k)} + A_{23}^{(k)}e_{33}^{(k)}$$
$$\sigma_{33}^{(k)} = A_{31}^{(k)}e_{11}^{(k)} + A_{32}^{(k)}e_{22}^{(k)} + A_{33}^{(k)}e_{33}^{(k)}$$
$$\sigma_{23}^{(k)} = 2A_{44}^{(k)}e_{23}^{(k)}, \quad \sigma_{13}^{(k)} = 2A_{55}e_{13}^{(k)}, \quad \sigma_{12}^{(k)} = 2A_{66}e_{12}^{(k)} \tag{73}$$

where the elastic constants of the kth layer are

$$A_{11}^{(k)} = A_{22}^{(k)} = \frac{\Delta_{11}^{(k)}}{\Delta_k}, \quad A_{12}^{(k)} = A_{21}^{(k)} = \frac{\Delta_{12}^{(k)}}{\Delta_k}$$

$$A_{13}^{(k)} = A_{31}^{(k)} = A_{23}^{(k)} = A_{32}^{(k)} = \frac{\Delta_{13}^{(k)}}{\Delta_k}, \quad A_{33}^{(k)} = \frac{\Delta_{33}^{(k)}}{\Delta_k}$$

$$\Delta_k = \frac{(1+\nu_k)[1-\nu_k-2(\nu'_k)^2 E_k/E'_k]}{E_k^2 E'_k}, \quad \Delta_{11}^{(k)} = \frac{1-(\nu'_k)^2 E_k/E'_k}{E_k E'_k}$$

$$\Delta_{12}^{(k)} = \frac{\nu_k+(\nu'_k)^2 E_k/E'_k}{E_k E'_k}, \quad \Delta_{13}^{(k)} = \frac{\nu'_k(1+\nu_k)}{E_k E'_k}, \quad \Delta_{33}^{(k)} = \frac{1-\nu_k^2}{E_k^2}$$

$$A_{44}^{(k)} = G_{23}^{(k)} = G'_k, \quad A_{55}^{(k)} = G_{13}^{(k)} = G'_k, \quad A_{66}^{(k)} = G_{12}^{(k)} = G_k. \tag{74}$$

Substituting the strains (68), (69) and (71) into (73) we obtain the tangential stresses

$$\begin{aligned}
\sigma_{11}^{(k)} &= A_{1i}^{(k)} e_{ir}^{(k)} + A_{i3}^{(k)} e_{33}^{(k)} \\
&= A_{1i}^{(k)}(v_{ig,r}\psi_g^{(k)} - \chi_{pg,ir}\psi_{pg}^{(k)} + k_{ir}\chi_{pg}\varphi_{pg}^{(k)}) + A_{13}^{(k)}\chi_{pg}\beta_{pg}^{(k)} \\
\sigma_{22}^{(k)} &= A_{2i}^{(k)} e_{ir}^{(k)} + A_{i3}^{(k)} e_{33} \\
&= A_{2i}^{(k)}(v_{ig,r}\psi_g^{(k)} - \chi_{pg,ir}\psi_{pg}^{(k)} + k_{ir}\chi_{pg}\varphi_{pg}^{(k)}) + A_{23}\chi_{pg}\beta_{pg}^{(k)} \\
i &= 1,2; \quad r = i; \quad p = 1,2,3; \quad g = 1,2,3,4
\end{aligned} \tag{75}$$

$$\begin{aligned}
\sigma_{12}^{(k)} &= 2A_{66}^{(k)} e_{12}^{(k)} \\
&= G_{12}^{(k)}[(v_{1g,2}+v_{2g,1})\psi_g^{(k)} - 2\chi_{pg,12}\psi_{pg}^{(k)} + 2k_{12}\chi_{pg}\varphi_{pg}^{(k)}] \\
p &= 1,2,3; \quad g = 1,2,3,4
\end{aligned} \tag{76}$$

and the stresses in the transverse direction

$$\begin{aligned}
\sigma_{13}^{(k)} &= 2A_{55}^{(k)} e_{13}^{(k)} = 2G_{13}^{(k)}(v_{1g}\alpha_g^{(k)} + \chi_{pg,1}\alpha_{pg}^{(k)}) \\
\sigma_{23}^{(k)} &= 2A_{44}^{(k)} e_{23}^{(k)} = 2G_{23}^{(k)}(v_{2g}\alpha_g^{(k)} + \chi_{pg,2}\alpha_{pg}^{(k)})
\end{aligned} \tag{77}$$

$$\begin{aligned}
\sigma_{33}^{(k)} &= A_{3i}^{(k)} e_{ir}^{(k)} + A_{33}^{(k)} e_{33}^{(k)} \\
&= A_{3i}^{(k)}(v_{ig,r}\psi_g^{(k)} - \chi_{pg,ir}\psi_{pg}^{(k)} + k_{ir}\chi_{pg}\psi_{pg}^{(k)}) + A_{33}^{(k)}\chi_{pg}\beta_{pg}^{(k)} \\
i &= 1,2, \quad r = i, \quad p = 1,2,3, \quad g = 1,2,3,4.
\end{aligned} \tag{78}$$

The transverse normal stresses $\sigma_{33}^{(k)}$ given in eqn (78) do not satisfy the conditions (2)–(4). In order to satisfy these conditions we have to use the transverse normal stresses in the form of eqn (48) as

$$\sigma_{33}^{(k)} = \chi_{pg} F_{pg}^{(k)}, \quad p = 1,2,3; \quad g = 3,4 \tag{79}$$

where

$$F_{pg}^{(k)} = E'_k \beta_{pg}^{(k)}.$$

In this case eqns (15) will be satisfied exactly when $\nu' = 0$, otherwise they will be satisfied integrally for the whole thickness of the laminate.

The equations, which are given above, define all the components of the displacement vector and the stress–strain tensor at an arbitrary point in the kth layer and they form the nonclassical higher-order model of the stress and strain state of a dynamically loaded laminated shell which takes into account transverse shear and normal deformation. The refined model includes the independent unknown functions of the reference surface v_{ig}, χ_{pg} ($i = 1,2$; $p = 1,2,3$; $g = 1,2$), the known functions v_{ig}, χ_{pg} ($i = 1,2$; $p = 1,2,3$; $g = 3,4$) which depend on the deformations of the reference surface obtained using the classical theory and on the loading conditions on the external surfaces, and the known functions of the normal

z which involve through the thickness distribution functions. The distribution functions are defined in a form which facilitates the satisfaction of the conditions on the external surfaces and the continuity conditions in between the layers when the reference surface is positioned arbitrarily through the thickness of the shell. Clearly, the governing equations are independent of the thicknesses, stiffnesses and other properties of the layers. Moreover, using this model we can consider layers with elastic characteristics that are constant or variable through the thickness and thus the model is comprehensive with respect to the properties of the layers.

The important feature of the present model is the inclusion of the dynamic factors such as forces of inertia and rotary inertia at the initial stage of derivation when the kinematic hypotheses are formulated.

VARIATIONAL EQUATION, EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Variational equation

The equations of motion and the boundary conditions may be determined using the Reissner variational principle

$$\int_{t_1}^{t_2} [(\delta R - \delta K) - \delta H] \, \mathrm{d}t = 0, \tag{80}$$

where δR is the variation of the Reissner functional, δK the variation of the kinetic energy, and δH the variation of the work done by the external forces.

For a laminated shell the variation of the Reissner functional has the following form (Reissner, 1950):

$$\delta R = \iiint_V [\sigma_{rs}^{(k)} \delta e_{rs}^{(k)} + \delta\sigma_{11}^{(k)}(e_{11}^{(k)} - a_{1s}^{(k)}\sigma_{ts}^{(k)}) + \delta\sigma_{22}^{(k)}(e_{22}^{(k)} - a_{2s}^{(k)}\sigma_{ts}^{(k)}) + \delta\sigma_{33}^{(k)}(e_{33}^{(k)} - a_{3s}^{(k)}\sigma_{ts}^{(k)})$$
$$+ \delta\sigma_{23}^{(k)}(2e_{23}^{(k)} - a_{44}^{(k)}\sigma_{23}) + \delta\sigma_{13}^{(k)}(2e_{13}^{(k)} - a_{55}^{(k)}\sigma_{13}^{(k)}) + \delta\sigma_{12}^{(k)}(2e_{12}^{(k)} - a_{66}^{(k)}\sigma_{12}^{(k)})] \, \mathrm{d}V$$
$$r, s = 1, 2, 3; \quad t = s. \tag{81}$$

Firstly, let us consider the implications of the variation of this functional being zero, i.e. $\delta R = 0$. From the basic lemma of calculus of variations it follows that in this case each term in the variational equation (81) is equal to zero. Substituting expressions for the components of the stress and strain tensors into the multipliers of the variations, we are able to ascertain that the stresses $\sigma_{ij}^{(k)}, \sigma_{i3}^{(k)}$ $(i, j = 1,2)$ are zero. This implies that the equations of the Hooke's law for the strains $e_{ij}^{(k)}, e_{i3}^{(k)}$ $(i, j = 1,2)$ are satisfied exactly.

The constitutive equations can be derived from the variational equation (80) in the form given by eqn (10). For the strains $e_{33}^{(k)}$, the constitutive equations are satisfied "integrally" (in the sense that the integral corresponding to this equation equals zero over the domain of the shell) since $\delta\sigma_{33}^{(k)} = 0$.

As long as all the terms in eqn (81), excluding the first, are identically equal to zero, the variation of the Reissner functional is equivalent to the variation of the potential energy of the deformation, namely

$$\delta R \to \delta U = \iiint_V \sigma_{rs}^{(k)} \delta e_{rs}^{(k)} \, \mathrm{d}V, \quad r, s = 1, 2, 3. \tag{82}$$

Variation of the potential energy

We now consider the tangential and normal components of the stress and strain tensors in eqn (82) given by

$$\delta U = \iiint_V [\sigma_{ij}^{(k)}\delta e_{ij}^{(k)} + 2\sigma_{i3}^{(k)}\delta e_{i3}^{(k)} + \sigma_{33}\delta e_{33}^{(k)}]\,\mathrm{d}V, \quad i,j = 1,2. \tag{83}$$

Substituting the strains from eqns (68), (69) and (71) into eqn (83), we can express the variation of the potential energy in terms of the displacements given by eqns (66) and (67). The through the thickness integral of the shell can be expressed as

$$\delta U = \iint_S \left\{ \int_{a_0}^{a_n} [\sigma_{ij}^{(k)}(\psi_g^{(k)}\delta v_{ig,j} - \psi_{pg}^{(k)}\delta\chi_{pg,ij} + k_{ij}\varphi_{pg}^{(k)}\delta\chi_{pg}) + \sigma_{i3}^{(k)}(\alpha_g^{(k)}\delta v_{ig} + \alpha_{pg}^{(k)}\delta\chi_{pg,i}) + \sigma_{33}^{(k)}(\beta_{pg}^{(k)}\delta\chi_{pg})]\,\mathrm{d}z \right\} \mathrm{d}S$$

$$i,j = 1,2; \quad p = 1,2,3; \quad g = 1,2,3,4, \tag{84}$$

where S is the two-dimensional domain of the shell surface. It is noted that the variations of the functions with subscripts $g = 3, 4$ are equal to zero. Using a notation similar to that of the classical theory, we may now consider the integral characteristics of the stresses due to internal forces and moments, namely generalized forces and moments which are defined as

$$[N_{ij}^{(q)}; M_{ij}^{(fq)}; N_{ij}^{(fq)}] = \int_{a_0}^{a_n} \sigma_{ij}^{(k)}[\psi_q^{(k)}; \psi_{fq}^{(k)}; \varphi_{fq}^{(k)}]\,\mathrm{d}z$$

$$[Q_i^{(q)}; Q_i^{(fq)}] = \int_{a_0}^{a_n} \sigma_{i3}^{(k)}[\alpha_q^{(k)}; \alpha_{fq}^{(k)}]\,\mathrm{d}z, \quad Q_3^{(fq)} = \int_{a_0}^{a_n} \sigma_{33}\beta_{fq}^{(k)}\,\mathrm{d}z$$

$$i,j = 1,2; \quad f = 1,2,3; \quad q = 1,2. \tag{85}$$

Substituting eqn (85) into eqn (84), and using Ostrogradsky–Gauss theorem, we obtain the following expression for the variation of the potential energy:

$$\delta U = -\iint_S [(N_{ij,j}^{(q)} - Q_i^{(q)})\delta v_{iq} + (M_{ij,ij}^{(fq)} - k_{ij}N_{ij}^{(fq)} + Q_{i,i}^{(fq)} - Q_3^{(fq)})\delta\chi_{fq}]\,\mathrm{d}S$$

$$+ \int_L [(N_{hh}^{(q)}\delta v_{hq} + N_{hl}^{(q)}\delta v_{lq}) + (M_{hh,h}^{(fq)} + 2M_{hl,l}^{(fq)} + Q_h^{(fq)})\delta\chi_{fq} - M_{hh}^{(fq)}\delta\chi_{fq,h}]\,\mathrm{d}L + [M_{hl}^{(fq)}\delta\chi_{fq}]_{L_1}^{L_2}, \tag{86}$$

where h and l are normal and tangent to the boundary L of the shell, respectively. For the forces on the boundary of the shell it was assumed that h and l are equivalent to i and j in eqn (85).

Variation of the kinetic energy

The expression for the variation of the kinetic energy may be written as

$$\delta K = -\iiint_V \rho_k \ddot{u}_s^{(k)}\,\delta u_s^{(k)}\,\mathrm{d}V = -\iiint_V \rho_k[\ddot{u}_i^{(k)}\,\delta u_i^{(k)} + \ddot{u}_3^{(k)}\,\delta u_3^{(k)}]\,\mathrm{d}V$$

$$= -\iiint_V \rho_k[\ddot{u}_i^{(k)}(\psi_q^{(k)}\delta v_{iq} - \psi_{fq}^{(k)}\delta\chi_{fq,i}) + \ddot{u}_3^{(k)}(\varphi_{fq}^{(k)}\delta\chi_{fq})]\,\mathrm{d}V$$

$$i = 1,2; \quad f = 1,2,3; \quad q = 1,2. \tag{87}$$

Let us introduce the integral characteristics of the inertia forces in the shell, i.e. the generalized inertia forces which are defined as

$$[T_i^{(q)}; T_i^{(fq)}] = \int_{a_0}^{a_n} \rho_k \ddot{u}_i^{(k)} [\psi_q^{(k)}; \psi_{fq}^{(k)}] \, dz$$

$$T_3^{(fq)} = \int_{a_0}^{a_n} \rho_k \ddot{u}_3^{(k)} \varphi_{fq}^{(k)} \, dz, \quad i = 1,2. \tag{88}$$

Substituting eqns (88) into eqn (87), we can rewrite the expression for the kinetic energy in the following form

$$\begin{aligned} \delta K &= -\iint_S [T_i^{(q)} \delta v_{iq} - T_i^{(fq)} \delta \chi_{fq,i} + T_3^{(fq)} \delta \chi_{fq}] \, dS \\ &= \iint_S [T_i^{(q)} \delta v_{iq} + (T_{i,i}^{(fq)} + T_3^{(fq)}) \delta \chi_{fq}] \, dS - \int_L (T_h^{(fq)} \delta \chi_{fq}) \, dL \\ & i = 1,2; \quad h = i; \quad f = 1,2,3; \quad q = 1,2. \end{aligned} \tag{89}$$

Variation of the work of the external loading

The variation of the work of the external loading is

$$\delta H = \delta H_1 + \delta H_2 \tag{90}$$

which consists of the work done by forces H_1 on the inner and outer surfaces and by the boundary forces H_2. Therefore, using the relations (66) and (67) and introducing the notation

$$p_i^{(q)} = p_i^- \psi_q^{(1)}(a_0) + p_i^+ \psi_q^{(n)}(a_n), \quad p_h^{(fq)} = p_h^- \varphi_{fq}^{(1)}(a_0) + p_h^+ \varphi_{fq}^{(n)}(a_n)$$

$$p_3^{(fq)} = [p_{i,i}^- \psi_{fq}^{(1)}(a_0) + p_3^- \varphi_{fq}^{(1)}(a_0) + p_{i,i}^+ \psi_{fq}^{(n)}(a_n) + p_3^+ \varphi_{fq}^{(n)}(a_n)] \tag{91}$$

for the generalized loading, we obtain the variation of the work of the load H_1 as

$$\begin{aligned} \delta H_1 &= \iint_S (p_s^- \delta u_s^{(1)} + p_s^+ \delta u_s^{(n)}) \, dS \\ &= \iint_S [p_i^- \delta u_i^{(1)} + p_i^+ \delta u_i^{(n)} + p_3^- \delta u_3^{(1)} + p_3^+ \delta u_3^{(n)}] \, dS \\ &= \iint_S \{p_i^- [\psi_q^{(1)}(a_0) \delta v_{iq} - \psi_{fq}^{(1)}(a_0) \delta \chi_{fq,i}] + p_3^- [\varphi_{fq}^{(1)}(a_0) \delta \chi_{fq}] \\ &\quad + p_i^+ [\psi_q^{(n)}(a_n) \delta v_{iq} - \psi_{fq}^{(n)}(a_n) \delta \chi_{fq,i}] + p_3^+ [\varphi_{fq}^{(h)}(a_n) \delta \chi_{fq}]\} \, dS \\ &= \iint_S (p_i^{(q)} \delta v_{iq} + p_3^{(fq)} \chi_{fq}) \, dS - \int_L (p_h^{(fq)} \delta \chi_{fq}) \, dL \\ & i = 1,2; \quad f = 1,2,3; \quad q = 1,2. \end{aligned} \tag{92}$$

Variation of the work of the boundary forces

The corresponding expression for the boundary forces has the form

$$\delta H_2 = \int_L \left\{ \int_{a_0}^{a_n} [\sigma_{hh}^{(k)} \delta u_h^{(k)} + \sigma_{h3}^{(k)} \delta u_3^{(k)} + \sigma_{hl}^{(k)} \delta u_l^{(k)}] \, dz \right\} dL, \tag{93}$$

where $\sigma_{hh}^{(k)}, \sigma_{h3}^{(k)}, \sigma_{hl}^{(k)}$ are components of the stress tensor and $u_h^{(k)}, u_3^{(k)}, u_l^{(k)}$ are components of the displacement vector at an arbitrary point of the kth layer on the boundary L of the shell. Taking into account that $i = h$ or l in the expressions (66) and (67) for the tangential and normal displacements and substituting (66) and (67) into eqn (93), we obtain

$$\delta H_2 = \int_L \left\{ \int_{a_0}^{a_n} [\sigma_{hh}^{(k)}(\varphi_q^{(k)}\delta v_{hq} - \psi_{fq}^{(k)}\delta\chi_{fq,h}) + \sigma_{hl}^{(k)}(\psi_q^{(k)}\delta v_{lq} - \psi_{fq}^{(k)}\delta\chi_{fq,l}) + \sigma_{h3}^{(k)}(\varphi_{fq}^{(k)}\delta\chi_{fq})]\,dz \right\} dL$$

$$= \int_L [\overset{*}{N}{}_{hh}^{(q)}\delta v_{hq} + \overset{*}{N}{}_{hl}^{(q)}\delta v_{lq} - \overset{*}{M}{}_{hh}^{(fq)}\delta\chi_{fq,h} + (\overset{*}{M}{}_{hl,l}^{(fq)} + \overset{*}{Q}{}_{h3}^{(fq)})\delta\chi_{fq}]\,dL - [\overset{*}{M}{}_{hl}^{(fq)}\delta\chi_{fq}]_{L_1}^{L_2}$$

$$f = 1,2,3; \quad q = 1,2, \tag{94}$$

where an asterisk denotes the forces acting on the boundary of the shell which may be expressed by eqns (85). Also in eqn (94) we have

$$\overset{*}{Q}{}_{h3}^{(fq)} = \int_{a_0}^{a_n} \sigma_{h3}^{(k)}\varphi_{fq}^{(k)}\,dz. \tag{95}$$

Equation of motion and boundary conditions

Substituting the variations (81), (86), (89), (92), (94) into (80), we derive the following variational equation:

$$\int_{t_1}^{t_2}\iint_S \{(N_{ij,j}^{(q)} - Q_i^{(q)} - T_i^{(q)} + p_i^{(q)})\delta v_{iq} + [M_{ij,ij}^{(fq)} - k_{ij}N_{ij}^{(fq)} + Q_{i,i}^{(fq)} - Q_3^{(fq)} - (T_{i,i}^{(fq)} + T_3^{(fq)})$$

$$+ p_3^{(fq)}]\delta\chi_{fq}\}\,dS\,dt - \int_{t_1}^{t_2}\int_L \{(N_{hh}^{(q)} - \overset{*}{N}{}_{hh}^{(q)})\delta v_{hq} + (N_{hl}^{(q)} - \overset{*}{N}{}_{hl}^{(q)})\delta v_{lq} + [M_{hh,h}^{(fq)} + 2M_{hl,l}^{(fq)} + Q_h^{(fq)} + Q_h^{(fq)}$$

$$+ T_h^{(fq)} + p_h^{(fq)} - (\overset{*}{M}{}_{hl,l}^{(fq)} + \overset{*}{Q}{}_{h3}^{(fq)})]\delta\chi_{fq} - (M_{hh}^{(fq)} - \overset{*}{M}{}_{hh}^{(fq)})\delta\chi_{fq,h}\}\,dL\,dt - [(M_{hl}^{(fq)} - \overset{*}{M}{}_{hl}^{(fq)})\delta\chi_{fq}]_{L_1}^{L_2} = 0$$

$$i,j = 1,2; \quad f = 1,2,3; \quad q = 1,2, \tag{96}$$

where t_1 and t_2 denote the initial and final time, respectively. The variations of the independent functions v_{iq} and χ_{fq}, which determine the displacements in the shell, have arbitrary values everywhere over the domain of the shell excluding the boundary and, consequently, they cannot be equal to zero. Equating the multipliers of the variations in the first integral of eqn (96) to zero, we obtain the system of equations of motion of the shell as

$$N_{ij,j}^{(q)} - Q_i^{(q)} - T_i^{(q)} + p_i^{(q)} = 0$$

$$M_{ij,ij}^{(fq)} - k_{ij}N_{ij}^{(fq)} + Q_{i,i}^{(fq)} - Q_3^{(fq)} - (T_{i,i}^{(fq)} + T_3^{(fq)}) + p_3^{(fq)} = 0$$

$$i,j = 1,2; \quad f = 1,2,3; \quad q = 1,2. \tag{97}$$

The boundary conditions follow from the boundary integral in eqn (96) and they may be written as

$$(N_{hh}^{(q)} - \overset{*}{N}{}_{hh}^{(q)})\delta v_{hq} = 0, \quad (N_{hl}^{(q)} - \overset{*}{N}{}_{hl}^{(q)})\delta v_{lq} = 0$$

$$[M_{hh,h}^{(fq)} + 2M_{hl,l}^{(fq)} + Q_h^{(fq)} + T_h^{(fq)} + p_h^{(fq)} - (\overset{*}{M}{}_{hl,l}^{(fq)} + \overset{*}{Q}{}_{h3}^{(fq)})]\delta\chi_{fq} = 0, \quad (M_{hh}^{(fq)} - \overset{*}{M}{}_{hh}^{(fq)})\delta\chi_{fq,h} = 0$$

$$f = 1,2,3; \quad q = 1,2. \tag{98}$$

There are 16 boundary conditions, which is the same as the order of the system of eqns (97). A detailed interpretation of the boundary conditions is given in Piskunov *et al.* (1987, 1993).

GENERALIZED FORCES AND MOMENTS AND THE SYSTEM OF GOVERNING EQUATIONS

Let us rewrite the generalized forces introduced earlier in eqns (85) and use the expressions for the stresses given by eqns (75), (77) and (78). Then we have for the tangential forces

$$
\begin{aligned}
N_{11}^{(q)} &= B_{1i}^{(gq)} v_{ig,r} - B_{1i}^{(pgq)} \chi_{pg,ir} + (C_{1i}^{(pgq)} k_{ir} + C_{1}^{(pgq)}) \chi_{pg} \\
N_{22}^{(q)} &= B_{2i}^{(gq)} v_{ig,r} - B_{2i}^{(pgq)} \chi_{pg,ir} + (C_{2i}^{(pgq)} k_{ir} + C_{2}^{(pgq)}) \chi_{pg} \\
N_{12}^{(q)} &= B^{(gq)} (v_{1g,2} + v_{2g,1}) - 2B^{(pgq)} \chi_{pg,12} + 2C^{(pgq)} k_{12} \chi_{pg} \\
& i = 1,2; \quad r = i,
\end{aligned} \tag{99}
$$

for the moments

$$
\begin{aligned}
M_{11}^{(fq)} &= \bar{B}_{1i}^{(gfq)} v_{ig,r} - D_{1i}^{(pgfq)} \chi_{pg,ir} + (E_{1i}^{(pgfq)} k_{ir} + E_{1}^{(pgfq)}) \chi_{pg} \\
M_{22}^{(fq)} &= \bar{B}_{2i}^{(gfq)} v_{ig,r} - D_{2i}^{(pgfq)} \chi_{pg,ir} + (E_{2i}^{(pgfq)} k_{ir} + E_{2}^{(pgfq)}) \chi_{pg} \\
M_{12}^{(fq)} &= \bar{B}^{(gfq)} (v_{1g,2} + v_{2g,1}) - 2D^{(pgfq)} \chi_{pg,12} + 2E^{(pgfq)} k_{12} \chi_{pg} \\
& i = 1,2; \quad r = i,
\end{aligned} \tag{100}
$$

for the higher-order tangential forces

$$
\begin{aligned}
N_{11}^{(fq)} &= \bar{C}_{1i}^{(gfq)} v_{ig,r} - \bar{E}_{1i}^{(pgfq)} \chi_{pg,ir} + (K_{1i}^{(pgfq)} k_{ir} + K_{1}^{(pgfq)}) \chi_{pg} \\
N_{22}^{(fq)} &= \bar{C}_{2i}^{(gfq)} v_{ig,r} - \bar{E}_{1i}^{(pgfq)} \chi_{pg,ir} + (K_{2i}^{(pgfq)} k_{ir} + K_{2}^{(pgfq)}) \chi_{pg} \\
N_{12}^{(fq)} &= \bar{C}^{(gfq)} (v_{1g,2} + v_{2g,1}) - 2\bar{E}^{(pgfq)} \chi_{pg,12} + 2K^{(pgfq)} k_{12} \chi_{pg} \\
& i = 1,2; \quad r = i,
\end{aligned} \tag{101}
$$

and for the shear forces

$$
\begin{aligned}
Q_{1}^{(q)} &= 2R_{1}^{(gq)} v_{1g} + 2R_{1}^{(pgq)} \chi_{pg,1}, \quad Q_{2}^{(q)} = 2R_{2}^{(gq)} v_{2g} + 2R_{2}^{(pgq)} \chi_{pg,2} \\
Q_{1}^{(fq)} &= 2\bar{R}_{1}^{(gfq)} v_{1g} + 2R_{1}^{(pgfq)} \chi_{pg,1}, \quad Q_{2}^{(fq)} = 2\bar{R}_{2}^{(gfq)} v_{2g} + 2R_{2}^{(pgfq)} \chi_{pg,2} \\
Q_{3}^{(fq)} &= \bar{C}_{i}^{(gfq)} v_{ig,r} - C_{3i}^{(pgfq)} \chi_{pg,ir} + (\bar{C}_{i}^{(pgfq)} k_{ir} + C_{3}^{(pgfq)}) \chi_{pg} \\
& i = 1,2; \quad r = i.
\end{aligned} \tag{102}
$$

In eqns (99)–(102) we have $q = 1,2$; $f, p = 1,2,3$; $g = 1,2,3,4$ and we also assume summation over i, p and g. The equations for the generalized forces and moments include the integrated stiffnesses of laminated shell given by

$$
\begin{aligned}
B_{\xi i}^{(gq)} &= \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_{g}^{(k)} \psi_{q}^{(k)} \, dz, \quad B_{\xi i}^{(pgq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_{pg}^{(k)} \psi_{q}^{(k)} \, dz \\
B^{(gq)} &= \int_{a_0}^{a_n} G_{12}^{(k)} \psi_{g}^{(k)} \psi_{q}^{(k)} \, dz, \quad B^{(pgq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \psi_{pg}^{(k)} \psi_{q}^{(k)} \, dz \\
\bar{B}_{\xi i}^{(gfq)} &= \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_{g}^{(k)} \psi_{fq}^{(k)} \, dz, \quad \bar{B}^{(gfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \psi_{g}^{(k)} \psi_{fq}^{(k)} \, dz \\
C_{\xi i}^{(pgq)} &= \int_{a_0}^{a_n} A_{\xi i}^{(k)} \varphi_{pg}^{(k)} \psi_{q}^{(k)} \, dz, \quad C^{(pgq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \varphi_{pg}^{(k)} \psi_{q}^{(k)} \, dz
\end{aligned}
$$

$$\bar{C}_{\xi i}^{(gfq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_g^{(k)} \varphi_{fq}^{(k)} \, dz, \quad \bar{C}^{(gfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \psi_g^{(k)} \varphi_{fq}^{(k)} \, dz$$

$$C_i^{(pgq)} = \int_{a_0}^{a_n} A_{i3}^{(k)} \beta_{pg}^{(k)} \psi_q^{(k)} \, dz, \quad \bar{C}_i^{(gfq)} = \int_{a_0}^{a_n} A_{3i}^{(k)} \psi_g^{(k)} \beta_{fq}^{(k)} \, dz$$

$$D_{\xi i}^{(pgfq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_{pg}^{(k)} \psi_{fq}^{(k)} \, dz, \quad D^{(pgfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \psi_{pg}^{(k)} \psi_{fq}^{(k)} \, dz$$

$$E_{\xi i}^{(pgfq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \varphi_{pg}^{(k)} \psi_{fq}^{(k)} \, dz, \quad E^{(pgfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \varphi_{pg}^{(k)} \psi_{fq}^{(k)} \, dz$$

$$\bar{E}^{(pgfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \psi_{pg}^{(k)} \varphi_{fq}^{(k)} \, dz, \quad \bar{E}_{\xi i}^{(pgfq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_{pg}^{(k)} \varphi_{fq}^{(k)} \, dz$$

$$E_i^{(pgfq)} = \int_{a_0}^{a_n} A_{i3}^{(k)} \beta_{pg}^{(k)} \psi_{fq}^{(k)} \, dz, \quad K_{\xi i}^{(pgfq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \varphi_{pg}^{(k)} \varphi_{fq}^{(k)} \, dz$$

$$K_i^{(pgfq)} = \int_{a_0}^{a_n} A_{i3}^{(k)} \beta_{pg}^{(k)} \varphi_{fq}^{(k)} \, dz, \quad K^{(pgfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \alpha_{pg}^{(k)} \alpha_{fq}^{(k)} \, dz$$

$$R_i^{(gq)} = \int_{a_0}^{a_n} G_{i3}^{(k)} \alpha_g^{(k)} \alpha_q^{(k)} \, dz, \quad R_i^{(pgq)} = \int_{a_0}^{a_n} G_{i3}^{(k)} \alpha_{pg}^{(k)} \alpha_q^{(k)} \, dz$$

$$R_i^{(pgfq)} = \int_{a_0}^{a_n} G_{i3}^{(k)} \alpha_{pg}^{(k)} \alpha_{fq}^{(k)} \, dz, \quad \bar{R}_i^{(gfq)} = \int_{a_0}^{a_n} G_{i3}^{(k)} \alpha_g^{(k)} \alpha_{fq}^{(k)} \, dz$$

$$C_{3i}^{(pgfq)} = \int_{a_0}^{a_n} A_{3i}^{(k)} \psi_{pg}^{(k)} \beta_{fq}^{(k)} \, dz, \quad C_i^{(pgfq)} = \int_{a_0}^{a_n} A_{3i}^{(k)} \varphi_{pg}^{(k)} \beta_{fq}^{(k)} \, dz$$

$$C_3^{(pgfq)} = \int_{a_0}^{a_n} A_{33}^{(k)} \beta_{pg}^{(k)} \beta_{fq}^{(k)} \, dz$$

$$i,q = 1,2; \quad p,f = 1,2,3; \quad g = 1,2,3,4; \quad \xi = 1,2. \tag{103}$$

The generalized forces of inertia (88) may be rewritten in the form

$$T_i^{(q)} = I^{(gq)} \ddot{v}_{ig} - I_1^{(pgq)} \ddot{\chi}_{pg,i}$$

$$T_i^{(fq)} = \bar{I}_1^{(gfq)} \ddot{v}_{ig} - I_2^{(pgfq)} \ddot{\chi}_{pg,i}$$

$$i = j = 1,2$$

$$T_3^{(fq)} = I_3^{(pgfq)} \ddot{\chi}_{pg}. \tag{104}$$

The equations for generalized forces of inertia include integrated density characteristics of the shell given by

$$I^{(gq)} = \int_{a_0}^{a_n} \rho_k \psi_g^{(k)} \psi_q^{(k)} \, dz, \quad I_1^{(pgq)} = \int_{a_0}^{a_n} \rho_k \psi_{pg}^{(k)} \psi_q^{(k)} \, dz$$

$$I_2^{(pgfq)} = \int_{a_0}^{a_n} \rho_k \psi_{pg}^{(k)} \psi_{fq}^{(k)} \, dz, \quad I_3^{(pgfq)} = \int_{a_0}^{a_n} \rho_k \varphi_{pg}^{(k)} \psi_{fq}^{(k)} \, dz$$

$$\bar{I}_1^{(gfq)} = \int_{a_0}^{a_n} \rho_k \psi_g^{(k)} \psi_{fq}^{(k)} \, \mathrm{d}z$$

$$q = 1,2; \quad p,f = 1,2,3; \quad g = 1,2,3,4. \tag{105}$$

The equations of the higher-order theory given above are in the form of a system of differential equations. This system may be written in the following matrix form:

$$[D]\{V\} + [I]\{\ddot{V}\} = [F]\{p\}, \tag{106}$$

where $[D]$ is the matrix of differential operators over the vector of unknown functions of the reference surface and time given by

$$\{V\} = \{v_{ig}; \chi_{pg}\}^{\mathrm{T}}, \quad i = 1,2; \quad p = 1,2,3; \quad g = 1,2,3,4. \tag{107}$$

$[I]$ is the matrix of the differential operators over the acceleration vector of these functions defined as

$$\{\ddot{V}\} = \{\ddot{v}_{ig}; \ddot{v}_{pg}\}^{\mathrm{T}}, \quad i = 1,2; \quad p = 1,2,3; \quad g = 1,2,3,4, \tag{108}$$

and $[F]$ is the matrix of differential operators over the vector of given loads which is

$$\{p\} = \{p_i^{\pm}; p_3^{\pm}\}^{\mathrm{T}}, \quad i = 1,2. \tag{109}$$

Finally, the matrices and the corresponding vectors may be written in the form

$$[D]\{V\} = \begin{bmatrix} v_{1g} & v_{2g} & \chi_{pg} \\ A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \tag{110}$$

where

$$[A_{11}] = B_{11}^{(gq)}(\ldots)_{,11} + B^{(gq)}(\ldots)_{,22} - 2R_1^{(gq)}(\ldots)$$

$$[A_{21}] = (B_{21}^{(gq)} + B^{(gq)})(\ldots)_{,21}$$

$$[A_{31}] = \{\bar{B}_{11}^{(gfq)}(\ldots)_{,11} + (\bar{B}_{21}^{(gfq)} + 2\bar{B}^{(gfq)})(\ldots)_{,22} - [(2\bar{R}_1^{(gfq)} - \bar{C}_1^{(gfq)})$$
$$- (\bar{C}_{11}^{(gfq)} k_{11} + \bar{C}_{21}^{(gfq)} k_{22})](\ldots)\}_{,1} - 2\bar{C}^{(gfq)} k_{21}(\ldots)_{,2}$$

$$[A_{12}] = (B_{12}^{(gq)} + B^{(gq)})(\ldots)_{,12}$$

$$[A_{22}] = (B_{22}^{(gq)}(\ldots)_{,22} + B^{(gq)}(\ldots)_{,11}) - 2R_2^{(gq)}(\ldots)$$

$$[A_{32}] = \{\bar{B}_{22}^{(gfq)}(\ldots)_{,22} + (\bar{B}_{12}^{(gfq)} + 2\bar{B}^{(gfq)})(\ldots)_{,11} + [(2\bar{R}_2^{(gfq)} - \bar{C}_2^{(gfq)})$$
$$- (\bar{C}_{22}^{(gfq)} k_{22} + \bar{C}_{12}^{(gfq)} k_{11})](\ldots)\}_{,2} + \bar{C}^{(gfq)} k_{21}(\ldots)_{,1}$$

$$[A_{13}] = -\{B_{11}^{(pgq)}(\ldots)_{,11} + (B_{12}^{(pgq)} + 2B^{(pgq)})(\ldots)_{,22} + [(2R_1^{(pgq)} - C_1^{(pgq)})$$
$$- (C_{11}^{(pgq)} k_{11} + C_{12}^{(pgq)} k_{22})](\ldots)\}_{,1} + 2C^{(pgq)} k_{12}(\ldots)_{,2}$$

$$[A_{23}] = -\{B_{22}^{(pgq)}(\ldots)_{,22} + (B_{21}^{(pgq)} + 2B^{(pgq)})(\ldots)_{,11} + [(2R_2^{(pgq)} - C_2^{(pgq)})$$
$$- (C_{22}^{(pgq)} k_{22} + C_{21}^{(pgq)} k_{11})](\ldots)\}_{,2} + 2C^{(pgq)} k_{21}(\ldots)_{,1}$$

Table 1. Some particular cases of the system of equations of motion for the laminated shell

Material assumptions	Present setting of problem (including inertia forces) Number of equations	Order of equations	Number of boundary conditions	Quasi-static setting of problem (excluding inertia forces) Number of equations	Order of equations	Number of boundary conditions
None	10	32	16	5	16	8
Excluding Poisson effect $\nu'_k = 0$	9	28	14	4	12	6
Excluding transverse shear $G'_k = \infty$	8	28	14	5	16	8
Excluding normal compression $E'_k = \infty$	7	20	10	4	12	6
Excluding shear and compression $G'_k = \infty, E'_k = \infty$	3	8	4	3	8	4

$$
\begin{aligned}
[A_{33}] = & -\{[D_{11}^{(pgfq)}(\ldots)_{,11} + (D_{12}^{(pgfq)} + 2D^{(pgfq)})(\ldots)_{,22}]_{,11} \\
& + [D_{22}^{(pgfq)}(\ldots)_{,22} + (D_{21}^{(pgfq)} + 2D^{(pgfq)})(\ldots)_{,11}]_{,22}\} \\
& - [2R_1^{(pgfq)} + (E_{11}^{(pgfq)} + \bar{E}_{11}^{(pgfq)})k_{11} + (E_{12}^{(pgfq)} + \bar{E}_{11}^{(pgfq)})k_{22} \\
& + (C_{31}^{(pgfq)} + E_1^{(pgfq)})](\ldots)_{,11} + [2R_2^{(pgfq)} + (E_{22}^{(pgfq)} + \bar{E}_{22}^{(pgfq)})k_{22} \\
& + (E_{21}^{(pgfq)} + \bar{E}_{21}^{(pgfq)})k_{11} + (C_{32}^{(pgfq)} + E_2^{(pgfq)})](\ldots)_{,22} \\
& + 4(E^{(pgfq)} + \bar{E}^{(pgfq)})k_{12}(\ldots)_{,12} - [C_3^{(pgfq)} + 4K^{(pgfq)}k_{12} \\
& + k_{11}(C_1^{(pgfq)} + K_{11}^{(pgfq)}k_{11} + K_{12}^{(pgfq)}k_{22} + K_1^{(pgfq)}) \\
& + k_{22}(C_2^{(pgfq)} + K_{22}^{(pgfq)}k_{22} + K_{21}^{(pgfq)}k_{11} + K_2^{(pgfq)})](\ldots).
\end{aligned}
$$

Moreover,

$$
[I]\{\ddot{V}\} = \begin{bmatrix} \ddot{v}_{1g} & \ddot{v}_{2g} & \ddot{\chi}_{1g} \\ -I^{gq} & — & I_1^{(pgq)}(\ldots)_{,1} \\ — & -I^{(gq)}(\ldots) & I_1^{(pgq)}(\ldots)_{,2} \\ -\bar{I}_1^{(pgq)}(\ldots)_{,1} & -\bar{I}_1^{(pgq)}(\ldots)_{,2} & (I_2^{(pgfq)}\nabla - I_3^{(pgfq)})(\ldots) \end{bmatrix} \tag{111}
$$

where ∇ is the Laplace operator.

$$
[F^-]\{p^-\} = \begin{bmatrix} p_1^- & p_2^- & p_3^- \\ \psi_q^{(1)}(a_0)(\ldots) & — & — \\ — & \psi_q^{(1)}(a_0)(\ldots) & — \\ \psi_{fq}^{(1)}(a_0)(\ldots)_{,1} & \psi_q^{(1)}(a_0)(\ldots)_{,2} & \varphi_{fq}^{(1)}(a_0)(\ldots) \end{bmatrix} \tag{112}
$$

$$
[F^+]\{p^+\} = \begin{bmatrix} p_1^+ & p_2^+ & p_3^+ \\ \psi_q^{(n)}(a_n)(\ldots) & — & — \\ — & \psi_q^{(n)}(a_n)(\ldots) & — \\ \psi_{fq}^{(n)}(a_n)(\ldots)_{,1} & \psi_{fq}^{(n)}(a_n)(\ldots)_{,2} & \varphi_{fq}^{(n)}(a_n)(\ldots) \end{bmatrix}. \tag{113}
$$

The total number of equations in the system (106) is equal to 10. All particular cases of the general system of eqns (106) can be obtained by making assumptions about the properties of the layers and these cases are shown in Table 1.

Considering some features of the present higher-order theory, one must note that the normal deformations depend on two factors which involve the influence of the Poisson's effect in the transverse direction, which can be ignored by assuming $\nu'_k = 0$, and the influence of the external loading and forces of inertia, which can be excluded by assuming $E'_k = \infty$.

The higher-order theory developed in the present study is considerably different from those in which the equations of motion are obtained on the basis of the quasi-static approach when the influence of the forces of inertia is not taken into account in the hypotheses.

SOME ANALYTICAL SOLUTIONS AND RESULTS

The analytical solution of the system of differential equations (106) is possible only for some particular cases. Let us consider the case of a simply supported shell with a rectangular plan view. The boundary conditions are specified as (e.g. for $x_1 = \text{const}$)

$$v_{2g} = 0, \quad N_{11}^{(q)} = 0, \quad \chi_{fg} = 0, \quad M_{11}^{(fq)} = 0$$
$$q = 1,2; \quad f = 1,2,3; \quad g = 1,2,3,4. \tag{114}$$

The solution may be obtained in a manner similar to the procedure used in the classical theory due to the fact that the equations of the higher-order theory have a mathematical structure which is similar to that of the classical shallow shell theory.

Let the loading be expressed by the trigonometric series

$$p_1^{\pm} = \sum_m \sum_n a_{mn}^{\pm} \cos \lambda_m x_1 \sin \gamma_n x_2 \, e^{-i\Omega_{mn}t}$$
$$p_2^{\pm} = \sum_m \sum_n b_{mn}^{\pm} \sin \lambda_m x_1 \cos \gamma_n x_2 \, e^{-i\Omega_{mn}t}$$
$$p_3^{\pm} = \sum_m \sum_n c_{mn}^{\pm} \sin \lambda_m x_1 \sin \gamma_n x_2 \, e^{-i\Omega_{mn}t}, \tag{115}$$

where

$$\lambda_m = \frac{m\pi}{a_1}, \quad \lambda_n = \frac{n\pi}{a_2}.$$

Now the unknown functions can be expanded as

$$v_{1g} = \sum_m \sum_n A_{mn}^{(g)} \cos \lambda_m x_1 \sin \gamma_n x_2 \, e^{-i\omega_{mn}t}$$
$$v_{2g} = \sum_m \sum_n B_{mn}^{(g)} \sin \lambda_m x_1 \cos \gamma_n x_2 \, e^{-i\omega_{mn}t}$$
$$\chi_{pg} = \sum \sum C_{mn}^{(pg)} \sin \lambda_m x_1 \sin \gamma_n x_2 \, e^{-i\omega_{mn}t}, \tag{116}$$

where a_1 and a_2 are the dimensions of the shell in the x_1 and x_2 directions, respectively; $a_{mn}^{\pm}, b_{mn}^{\pm}, c_{mn}^{\pm}$ are Fourier coefficients for the entries of the load vector; $A_{mn}^{(g)}, B_{mn}^{(g)}, C_{mn}^{(pg)}$ are amplitudes of the unknown functions; Ω_{mn} and ω_{mn} are the frequencies of the excitation load and free vibrations of the shell, respectively.

The solution of the forced vibration problem reduces to the solution of the system of linear algebraic equations

$$[D_0 - I_0\omega_{mn}^2]\{A_{mn}\} = [F_0\Omega_{mn}^2]\{a_{mn}\}, \tag{117}$$

where the vector of the amplitudes of the unknown functions is given by

$$\{A_{mn}\} = \{A_{mn}^{(g)}; B_{mn}^{(g)}; C_{mn}^{(pg)}\}^{\mathrm{T}}, \quad p = 1,2,3; \quad g = 1,2 \tag{118}$$

and the vector of the given load amplitudes by

$$\{a_{mn}\} = \{a_{mn}^{\pm}; b_{mn}^{\pm}; c_{mn}^{\pm}\}^{\mathrm{T}}. \tag{119}$$

The solution for the natural frequencies reduces to the solution of the generalized eigenvalue problem which may be expressed by the following characteristic determinant equation for the spectrum of the frequencies ω_{mn}:

$$|D_0 - I_0\omega_{mn}^2| = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = 0, \tag{120}$$

where

$$\begin{aligned}
[A_{11}] &= -[(B_{11}^{(gq)}\lambda_m^2 + B^{(gq)}\gamma_n^2) - 2R_1^{(gq)}] - I^{(gq)}\omega_{mn}^2 \\
[A_{21}] &= -(B_{21}^{(gq)} + B^{(gq)})\lambda_m\gamma_n \\
[A_{31}] &= \{\bar{B}_{11}^{(gfq)}\lambda_m^2 + (\bar{B}_{21}^{(gfq)} + 2\bar{B}^{(gfq)}\gamma_n^2) - [(2\bar{R}_1^{(gfq)} - \bar{C}_1^{(gfq)}) \\
&\quad -(\bar{C}_{11}^{(gfq)}k_{11} + \bar{C}_{21}^{(gfq)}k_{22})]\lambda_m\} + \bar{I}_1^{(gfq)}\lambda_m\omega_{mn}^2 \\
[A_{12}] &= -(B_{12}^{(gq)} + B^{(gq)})\lambda_m\gamma_n \\
[A_{22}] &= -[(B_{22}^{(gq)}\lambda_m^2 + B^{(gq)}\gamma_n^2) - 2R_2^{(gq)}] - I^{(gq)}\omega_{mn}^2 \\
[A_{32}] &= \{\bar{B}_{22}^{(gfq)}\gamma_n^2 + (\bar{B}_{12}^{(gfq)} + 2\bar{B}^{(gfq)})\lambda_m^2 - [(2\bar{R}_2^{(gfq)} - \bar{C}_2^{(gfq)}) \\
&\quad -(\bar{C}_{22}^{(gfq)}k_{22} + \bar{C}_{12}^{(gfq)}k_{11})]\gamma_n\} + \bar{I}_1^{(gfq)}\gamma_n\omega_{mn}^2 \\
[A_{13}] &= \{B_{11}^{(pgq)}\lambda_m^2 + (B_{12}^{(pgq)} + 2B^{(pgq)})\gamma_n^2 - [(2R_1^{(pgq)} - C_1^{(pgq)}) \\
&\quad -(C_{11}^{(pgq)}k_{11} + C_{12}^{(pgq)}k_{22})]\lambda_m\} + I_1^{(pgq)}\lambda_m\omega_{mn}^2 \\
[A_{23}] &= \{B_{22}^{(pgq)}\gamma_n^2 + (B_{21}^{(pgq)} + 2B^{(pgq)})\lambda_m^2 - [(2R_2^{(pgq)} - C_2^{(pgq)}) \\
&\quad -(C_{22}^{(pgq)}k_{22} + C_{21}^{(pgq)}k_{11})]\gamma_n\} + I_1^{(pgq)}\gamma_n\omega_{mn}^2 \\
[A_{33}] &= -\{[D_{11}^{(pgfq)}\lambda_m^2 + (D_{12}^{(pgfq)} + 2D^{(pgfq)})\gamma_n^2]\lambda_m^2 + [D_{22}^{(pgfq)}\gamma_n^2 + (D_{21}^{(pgfq)} + 2D^{(pgfq)})\lambda_m^2]\gamma_n^2\} \\
&\quad -2[R_1^{(pgfq)} + (E_{11}^{(pgfq)} + \bar{E}_{11}^{(pgfq)})k_{11} + (E_{12}^{(pgfq)} + \bar{E}_{12}^{(pgfq)})k_{22} + (C_{31}^{(pgfq)} + E_1^{(pgfq)})]\lambda_m^2 \\
&\quad -[2R_2^{(pgfq)} + (E_{22}^{(pgfq)} + \bar{E}_{22}^{(pgfq)})k_{22} + (E_{21}^{(pgfq)} + \bar{E}_{21}^{(pgfq)})k_{11} + (C_{32}^{(pgfq)} + E_2^{(pgfq)})]\gamma_n^2 \\
&\quad -[C_3^{(pgfq)} + k_{11}(C_1^{(pgfq)} + K_{11}^{(pgfq)}k_{11} + K_{12}^{(pgfq)}k_{22} + K_1^{(pgfq)}) \\
&\quad +k_{22}(C_2^{(pgfq)} + K_{22}^{(pgfq)}k_{22} + K_{21}^{(pgfq)}k_{11} + K_2^{(pgfq)})] - [I_2^{(pgfq)}(\lambda_m^2 + \gamma_n^2) + I_3^{(pgfq)}]\omega_{mn}^2.
\end{aligned}$$

Special cases of the general problem may be obtained by specifying the material properties. In these cases the order of the determinant is reduced and the corresponding number of equations for each case is given in Table 1. Next we consider some specific problems.

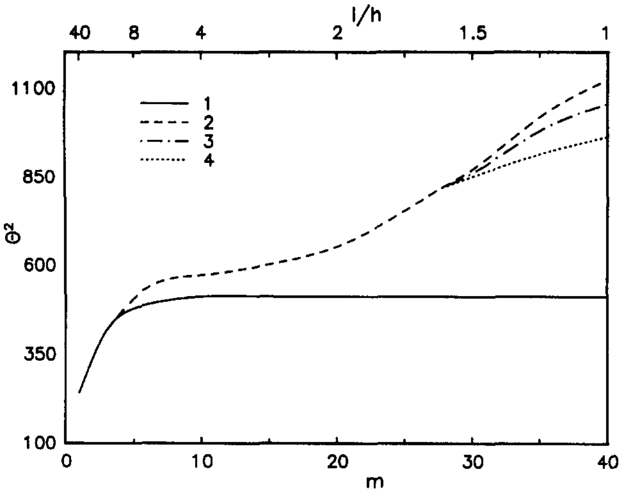


Fig. 2. Comparison of the solutions for the sandwich plates: (1) Reissner–Timoshenko theory; (2) present theory; (3) first-order theory; (4) exact three-dimensional solution.

Problem 1

Let us consider the free vibrations of a homogeneous square plate ($a_1 = a_2 = a$) with boundary conditions given by eqns (114). The results presented in Table 2 are obtained for frequencies which are equal to the half-wavelength l of the vibration mode in the orthogonal directions ($l = a/m$). Results are presented for various ratios of half-wavelength l and thickness of the plate h where the side length is taken as $a = 40h$. The results agree very closely with the exact three-dimensional solution in the interval $40 \geqslant l/h \geqslant 1$. Thus the higher-order theory offered allows the determination of frequencies for which the half-wavelength is equal to the thickness of the homogeneous plate ($l/h = 1$). The classical theory is acceptable only when $l/h > 8$.

Problem 2

Let us consider the problem of free vibrations of a square sandwich plate made of isotropic layers with $a_1 = a_2 = a$. The following characteristics are used:

$$2h_1 + h_2 = h, \quad h_2 = 18h_1, \quad a = 40h$$

$$G_1 = 10^3 G_2, \quad \rho_1 = 10\rho_2, \quad \nu_1 = 0.3, \quad \nu_2 = 0,$$

where h_1 and h_2 are the thicknesses of the surface and core layers, respectively. Figure 2 shows the curves of the normalized parameter

$$\theta^2 = 10^4 \frac{\omega^2 l^2}{\pi^2} \cdot \frac{\rho_2}{2G_2}$$

plotted against h_2/h_1, where ω is the fundamental vibration frequency. Solutions are given for the following theories:

Table 2. Free vibration frequencies of the homogeneous plate

Problem data: $\theta = 10^2 \frac{\omega l}{\pi} \sqrt{\rho/2G}$, $l = a/m$, $a = 40h$, $\nu = 0.3$

Theory	$l/h = 40$	8	4	2	1
Three-dimensional	5.408	25.74	45.62	68.25	84.22
Present HOT	5.408	25.74	45.62	68.29	84.41
Classical	5.419	27.10	54.19	—	—

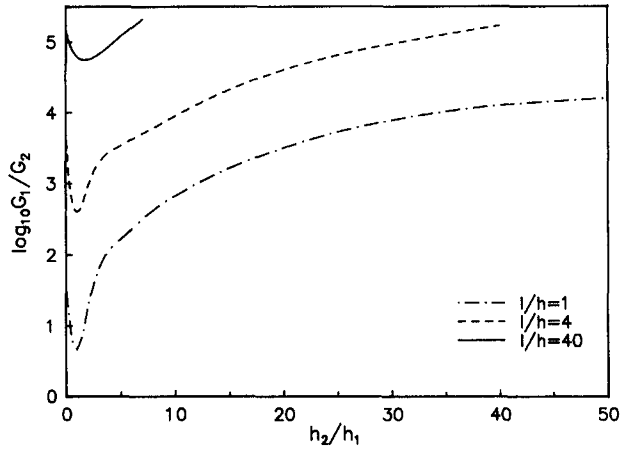


Fig. 3. Domains in which the higher-order theory and the three-dimensional theory coincide. (This domain lies under each curve.)

(1) Reissner–Timoshenko theory (involving the hypothesis of plane sections remaining plane);
(2) present higher-order theory;
(3) three-dimensional theory for the core and classical theory for the external layers;
(4) exact three-dimensional theory.

The results of the higher-order theory coincide very closely with those of the three-dimensional theory on the interval $40 \geqslant l/h \geqslant 1$. When $l/h = 1$, the discrepancy is 5% for the parameter θ. The solution based on the Reissner–Timoshenko theory cannot be used for the investigation of large frequencies when $l/h > 4$.

The limitations on the use of the higher-order theory for sandwich plates depend on three parameters: l/h ratio between the half-wavelength and the thickness; h_2/h_1 ratio between the thicknesses of the core and external layers and G_1/G_2 ratio between the shear moduli of the external and core layers. The regions of these ratios for which the higher-order theory coincides with the three-dimensional theory are shown in Fig. 3. The domain where the higher-order theory is applicable lies under the corresponding curve. It is observed that when $l/h = 40$ the results obtained on the basis of the higher-order theory occupy almost the entire region within the following bounds:

$$0 \leqslant \log(G_1/G_2) \leqslant 5, \quad 0 \leqslant h_2/h_1 \leqslant 50.$$

If $l/h \leqslant 4$ the bounds are

$$0 \leqslant \log(G_1/G_2) \leqslant 4, \quad 10 \leqslant h_2/h_1 \leqslant 50$$

and for $l/h = 1$

$$0 \leqslant \log(G_1/G_2) \leqslant 3, \quad 10 \leqslant h_2/h_1 \leqslant 50.$$

Within $0 \leqslant h_2/h_1 \leqslant 10$ the domain of the values $\log(G_1/G_2)$ is reduced on the average by one order. The highest reduction is reached when the thickness ratio is $h_2/h_1 = 0.6$. For this ratio we observe the highest influence of the transverse shear deformations for the sandwich plate.

Problem 3

In this problem the results obtained for the fundamental frequency of a sandwich beam are compared with the experimental data obtained by Khatua and Cheng (1973) and Osternik (1973). The results obtained using the higher-order theory are in good agreement

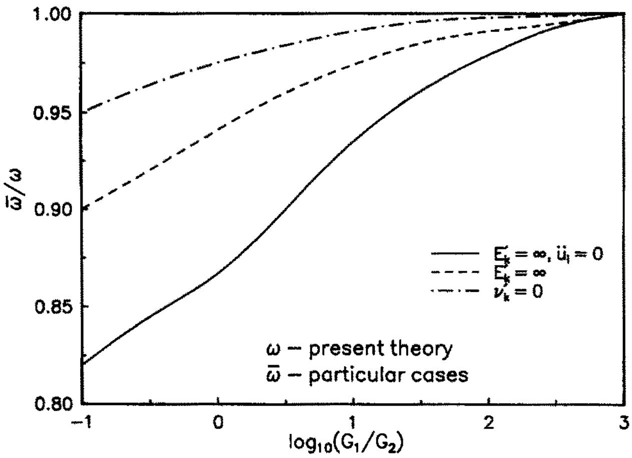


Fig. 4. Comparison of the present theory with theories neglecting the effects of some parameters.

with the experimental results as shown in Table 3. The classical theory is inapplicable when we either decrease the relative length of the span l/h or the shear modulus of the core.

Problem 4

Figure 4 shows the results of the comparison of natural frequencies for simply supported sandwich plates using different theories. In the figure ω denotes the fundamental frequencies obtained using the general solution (120) and $\bar{\omega}$ are the frequencies obtained using particular cases given in Table 1. Results are given for parameters

$$l/h = 4, \quad \rho_1/\rho_2 = 10^2 \quad \text{and} \quad h_2/h_1 = 18.$$

In the case when the major influence on the frequencies is due to the effect of the transverse shear deformations [$\log(G_1/G_2) \geqslant 3$], all results agree very closely. The influence of the normal deformations increases as the shear modulus of the core material becomes larger [$\log(G_1/G_2) < 3$]. The solutions become inaccurate for the case when dynamic factors in the hypotheses are excluded by setting $\ddot{u}_i = 0$ (quasi-static problem) and for the case when the normal deformation is not taken into account ($E'_k = \infty$).

Problem 5

Let us consider various plates and shallow shells with different sequences of the strong and weak layers through the thickness. The total thickness and mass of the shell for each case is kept constant. The fundamental frequencies ω_0 and their ratio to the frequencies ω, which are obtained using the classical theory, are given in Table 4. It is observed that the influence of the transverse shear becomes significant in structures where the strong layers are separated by weak layers. The influence of transverse shear and normal deformation increases as the strong material is redistributed more closely to the external surfaces, and also as the half-wavelength of the vibration is reduced. For single layer plates, the results are very close to those given by the classical theory.

Table 3. Comparison with experimental results for sandwich beams

h_2/h_1	l/h	$\log(G_1/G_2)$	Experimental ω (s^{-1})	HOT ω (s^{-1})	Δ (%)	Classical ω (s^{-1})	Δ (%)
7.8	42.4	1.43	70.2	70.7	0.7	72.6	3.4
5.3	54.9	2.65	80.1	78.3	−2.2	86.6	8.1
17.0	14.7	2.82	123	120	−2.3	214	74
7.9	11.8	2.82	161	151	−6.0	390	143

Table 4. Free vibration frequencies of plates of equal mass

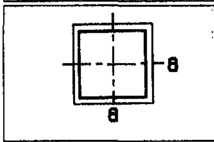
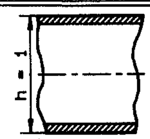
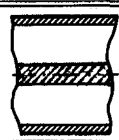
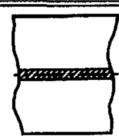
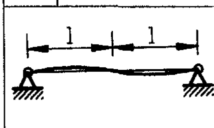
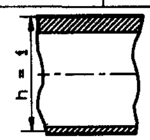
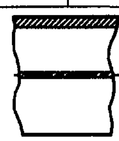
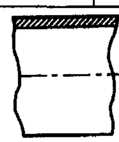
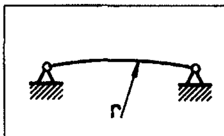
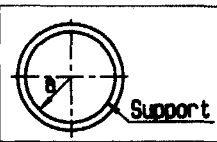
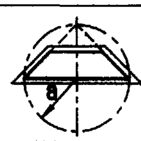
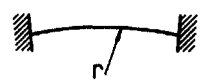
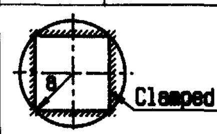
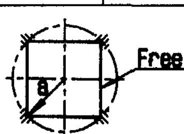
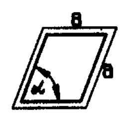
Problem data: $a = 40h$; $G_1 = 10^3 G_2$; $\rho_1 = 10\rho_2$; $\nu_1 = 0.3$; $\nu_2 = 0.4$							
Frequencies: $\omega/2\pi$, sec^{-1}. Shear influence: ω_0/ω							
Simply supported square plates of equal mass							
			0.025 0.9 0.025		0.025 0.45 0.05 0.45 0.025		0.45 0.1 0.45
l/h	40	260	1.36	216	1.20	31.1	1.0
	20	677	2.10	627	1.65	124	1.0
	10	1481	3.84	1421	2.81	488	1.02
			0.075 0.9 0.025		0.075 0.45 0.025 0.45		0.1 0.9
l/h	40	241	1.19	145	1.15	46.4	1.01
	20	657	1.96	444	1.51	183	1.02
	10	1461	3.38	1069	2.48	692	1.08

Table 5. The fundamental frequencies of sandwich plates and spherical shells

								
r/a	∞	129	1.17	398	1.59	552		1.87
	4	529	1.01	649	1.25	753		1.52
	2	1034	1.00	1100	1.08	1164		1.25
								
r/a	∞	452	1.44	154	1.26	$\alpha°$, 60	310	1.44
	4	684	1.21	525	1.03	45	396	1.59
	2	1120	1.08	1037	1.00	30	595	1.95

Problem 6

Next we consider sandwich plates and spherical shells with different boundary conditions and different shapes in plan. A number of analytical and numerical methods of solution is used to obtain the numerical results. Table 5 shows the fundamental vibration frequencies $\omega/2\pi$ s^{-1} in the sandwich structures which are circumscribed by circles of equal radii in plan. The influence of the transverse shear deformation (ratio ω_0/ω) becomes more pronounced as the area enclosed by the structure becomes smaller. This influence is also more pronounced for plates with clamped boundaries as compared to plates with simply supported or free boundaries. It is also observed that this influence decreases as the curvature of the shell gets larger and increases as the acute angle of the oblique-angled plate decreases.

Tables 4 and 5 indicate that as the influence of the transverse shear and normal deformation increases, the difference in the vibration frequencies of plates and shells with different geometrical and mechanical properties of layers and different boundary conditions decreases and the frequency spectrum broadens.

SUMMARY AND CONCLUSIONS

A higher-order theory of laminated plates and shells, which takes into account transverse shear and normal deformation, is developed for the solution of dynamic problems. The proposed theory is capable of treating plates and shells with an arbitrary number and sequences of layers which may differ significantly in their physical and mechanical properties. The elastic characteristics may be constant or variable through the thickness of each layer. The kinematic hypotheses are derived using an iterative technique where the classical theory is used as a first approximation. The important feature of the model is that the dynamic factors, such as forces of inertia and rotary inertia, are included in the model at the initial stage when the kinematic hypotheses are formulated. This procedure leads to a number of new unknown functions and subsequently to a number of additional higher-order equations of motion. The new variables which are introduced have clear physical meanings. The direct influence of the loading conditions on the transverse shear and normal deformation of the shell is also incorporated into the model. It is shown that the results obtained using a quasi-static theory, in which the forces of inertia and rotary inertia are neglected in the kinematic hypotheses, are not as accurate as the results obtained on the basis of the present approach. The level of accuracy of a given theory depends on several factors which are discussed and elucidated in the context of example problems.

The equations of motion and the complete set of boundary conditions are derived using a variational formulation. Different particular cases are also studied as special cases of the general theory.

The system of governing differential equations is derived and some analytical solutions are obtained. It is shown that if the layers are significantly different in their elastic properties, it is necessary to take into account both transverse shear and normal deformation in order to obtain accurate results. The results obtained compare favorably with three-dimensional elasticity solutions and with experimental data.

Summarizing the results presented in this paper, it is noted that the proposed theory can lead to a substantial improvement of the accuracy in the following problems:

(i) vibration processes with large frequencies;
(ii) wave processes;
(iii) non-stationary vibrations.

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